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ON SOME TOPOLOGICAL PROPERTIES OF THE CLASS OF NORMALIZED AND STARLIKE MAPS OF THE UNIT POLYDISK IN C^n

In this paper we consider univalent holomorphic maps of the unit polydisk P^n into $\mathbb{C}^n$. We find a necessary and sufficient condition for this function to be starlike. Further, we show that the class of normalized and starlike maps of the unit polydisk P^n into $\mathbb{C}^n$ is compact and connected.

Let $\mathbb{C}^n$ denote the space of n complex variables $z = (z_1, \ldots, z_n)$, $z_j \in \mathbb{C}$, $j = 1, \ldots, n$. For $(z_1, \ldots, z_n) = z \in \mathbb{C}^n$, define $\|z\| = \max_{1 \leqslant j \leqslant n} |z_j|$. Let $P_r^n = \{z \in \mathbb{C}^n;\ \|z\| < r\}$ and $P^n = P_1^n$. We shall denote by I the identity map on $\mathbb{C}^n$. The class of holomorphic maps of a domain Ω (contained in $\mathbb{C}^n$) into $\mathbb{C}^n$ is denoted by $H(\Omega)$. The class $H(\Omega)$ will be taken as a topological space with topology of almost uniform convergence (see [5], p. 66).

Let $M(P_r^n)$ be the class of maps $h : P_r^n \to \mathbb{C}^n$ which are holomorphic and satisfy the following conditions: $h(0) = 0$, $Dh(0) = I$ and $\mathrm{re}\,(h_j(z)/z_j) \geqslant 0$ when $\|z\| = |z_j| > 0$ $(1 \leqslant j \leqslant n)$, where $h = (h_1, \ldots, h_n)$ (see [8], [9]).

We say that $f \in H(P_r^n)$ is starlike if it is univalent $f(0) = 0$ and $f(P_r^n)$ is a starlike set (i.e. $(1 - t)f(P_r^n) \subset f(P_r^n)$ for $0 \leqslant t \leqslant 1$).

Let $\mathcal{S}_o(P_r^n)$ denote the class of starlike maps $f : P_r^n \to \mathbb{C}^n$ such that $Df(0) = I$.

The proofs of the main theorems in this paper are based on relations between the classes $\mathcal{G}_0(P^n)$ and $M(P^n)$.

With the above notation, we can write *theorem 1* from [8] in the form:

Theorem A. If $f \in \mathcal{G}_0(P^n)$, then there exists a function $h \in M(P^n)$ such that

$$f(z) = Df(z)h(z) \quad \text{for} \quad z \in P^n$$

A continuation of studies of these relations can be found in paper [6].

In our case (i.e. $X = \mathbb{C}^n$), every starlike map is biholomorphic hence theorems 3, 4 and 5 from [6] can be formulated in the following form:

Theorem B. If $h \in M(P^n)$, then the equality

$$Df(x)h(x) = f(x) \quad \text{for} \quad x \in P^n$$

where $f(0) = 0$, $Df(0) = I$, has a unique solution f which belongs to $\mathcal{G}_0(P^n)$.

At the beginning of this paper we are occupied in strengthening *theorem B.* The following lemma which is a generalization of *theorem 7* from [2] will be useful.

Lemma 1. Let $f \in H(P^n_r)$ be a locally biholomorphic map such that $f(0) = 0$ and $Df(0) = I$. Then f is a starlike map on P^n_r if and only if there exists a function $h \in M(P^n_r)$ such that

$$f(z) = Df(z)h(z) \quad \text{for} \quad z \in P^n_r$$

P r o o f. Suppose that $f \in H(P^n_r)$ is a locally biholomorphic map such that $f(0) = 0$, $Df(0) = I$ and $Df(z)h(z) = f(z)$ for $z \in P^n_r$, where $h \in M(P^n_r)$.

Now, we define the functions:

$$\tilde{f}(z) = \frac{1}{r}f(rz) \quad \text{and} \quad \tilde{h}(z) = \frac{1}{r}h(rz) \quad \text{for} \quad z \in P^n.$$

It is easy to see that $\tilde{f} : P^n \to \mathbb{C}^n$ is locally biholomorphic, $\tilde{f}(0) = 0$, $D\tilde{f}(0) = I$, while $\tilde{h} \in M(Pn)$. Since $f(rz) = Df(rz)h(rz)$ for $z \in P^n$, therefore $\tilde{f}(z) = D\tilde{f}(z)\tilde{h}(z)$ for $z \in P^n$. By *theorem 7* from [2], we obtain that $\tilde{f} \in \mathcal{G}_0(P^n)$, which implies $f \in \mathcal{G}_0(P^n_r)$.

Suppose now that $f \in \mathcal{G}_0(P^n_r)$. Let us consider, as previously, a map $\tilde{f}(z) = \frac{1}{r}f(rz)$ for $z \in P^n$. Such a map belongs to $\mathcal{G}_0(P^n)$, hence, by *theorem A,* there exists $\tilde{h} \in M(P^n)$ such that $\tilde{f}(z) = D\tilde{f}(z)\tilde{h}(z)$ for $z \in P^n$. Observe that $r\tilde{f}(\frac{z}{r}) = D\tilde{f}(\frac{z}{r})r\tilde{h}(\frac{z}{r})$ for

$z \in P_r^n$. Put $h(z) = r\tilde{h}(\frac{z}{r})$ for $z \in P_r^n$; then $h \in M(P_r^n)$ and the equation $f(z) = Df(z)h(z)$ for $z \in P_r^n$ is satisfied. This completes the proof.

Corollary. If $f \in \mathcal{G}_o(P^n)$, then $f(P_r^n)$ is a starlike set for any $r \in (0, 1)$.

Theorem 1. If $h \in M(P^n)$, then the equation

$$Df(z)h(z) = f(z) \quad \text{for} \quad z \in P^n \tag{1}$$

possesses exactly one solution $f \in H(P^n)$ such that $f(0) = 0$ and $Df(0) = I$. Moreover, $f \in \mathcal{G}_o(P^n)$.

P r o o f. Let $h \in M(P^n)$ and let $v = v(z, t)$ for $z \in P^n$ and for $t \geqslant 0$ be a solution of the equation

$$\frac{\partial v}{\partial t}(z, t) = -h(v(z, t)), \quad v(z, 0) = z \quad \text{for} \quad z \in P^n$$

(see *lemma 1* from [6]). Then, by *theorem 3* from [6], the function f defined as $f(z) = \lim_{t \to \infty} e^t v(z, t)$ for $z \in P^n$ belongs to $\mathcal{G}_o(P^n)$. Hence, in virtue of *theorem 4* from [6], we obtain that f fulfils equation (1). Suppose that there exists a map $g \in H(P^n)$ such that $g(0) = 0$, $Dg(0) = I$ and g satisfies equation (1). Since $Dg(0) = I$, therefore, in virtue of *theorem 1.3.7* from [7], there exists $\varrho (0 < \varrho < 1)$ such that $g|P_\varrho^n$ is a biholomorphic map. It follows from *lemma 1* that $g|P_\varrho^n$ is starlike.

Let us now consider a map $u : P_\varrho^n \times [0, \infty) \to C^n$ defined in the following way:

$$u(z, t) = g^{-1}(e^t g(z)) \quad \text{for} \quad z \in P_\varrho^n \text{ and } t \geqslant 0$$

It is not difficult to show that $\frac{\partial u}{\partial t}(z, t) = -h(u(z, t))$ for $z \in P_\varrho^n$ and $t \geqslant 0$.

Since $v = v(z, t)$ for $z \in P^n$ and $t \geqslant 0$ fulfils the above equation as well, the uniqueness of the solution of this equation (see *lemma 5* from [2]) implies that $v(z, t) = u(z, t)$ for $z \in P_\varrho^n$ and $t \geqslant 0$. It is easy to see that $\lim_{t \to \infty} e^t u(z, t) = g(z)$ for $z \in P_\varrho^n$. Since $f(z) = \lim_{t \to \infty} e^t v(z, t)$ for $z \in P_\varrho^n$ therefore $f(z) = g(z)$ for $z \in P_\varrho^n$. This and the analytic extension principle (see *theorem 9.4.2.* from [1]) imply that $f(z) = g(z)$ for $z \in P^n$, which ends the proof.

The next theorem will be preceded by two lemmas.

Lemma 2. If $h \in M(P^n)$, then

$$\|h(z)\| \leqslant \|z\| \frac{1 + \|z\|}{1 - \|z\|} \quad \text{for} \quad z \in P^n.$$

P r o o f. Let $h \in M(P^n)$. Denote $E_k^n = \{z \in P^n;\ \|z\| \leqslant |z_k|$ where $z = (z_1, \ldots, z_n)\}$ for $k = 1, \ldots, n$. Let k $(1 \leqslant k \leqslant n)$ be any fixed number and put $F_k(z) = \frac{h_k(z)}{z_k}$ for $z \in E_k^n - \{0\}$, where $z = (z_1, \ldots, z_n)$ and $h = (h_1, \ldots, h_n)$. It is obvious that $\operatorname{re} F_k(z) > 0$ for $z \in E_k^n - \{0\}$. Now, we define a function H_k in the following way:

$$H_k(t_1, \ldots, t_n) = F_k(t_1 t_k, \ldots, t_{k-1} t_k, t_k, t_{k+1} t_k, \ldots, t_n t_k)$$

for all $t = (t_1, \ldots, t_n) \in P^n$ such that $t_k \neq 0$. Since h_k is a holomorphic function on P^n and $Dh(0) = I$, therefore we can represent it in the form of the absolutely convergent power series

$$h_k(z) = z_k + \sum_{\substack{|\nu|>1 \\ \nu \in N^n}} \alpha_\nu^{(k)} z^\nu \quad \text{for} \quad z \in P^n$$

(see 9.3 from [1]). Using this representation, we obtain that

$$H_k(t) = 1 + \sum_{\substack{|\nu|>1 \\ \nu \in N^n}} \alpha_\nu^{(k)} t_1^{\nu_1} \ldots t_{k-1}^{\nu_{k-1}} t_{k+1}^{\nu_{k+1}} \ldots t_n^{\nu_n} t_k^{|\nu|-1},$$

where $\nu = (\nu_1, \ldots, \nu_n)$, for all $t = (t_1, \ldots, t_n) \in P^n$ such that $t_k \neq 0$.

Let us extend the function H_k to the entire polydisk P^n by putting, for $t = (t_1, \ldots, t_{k-1}, 0, t_{k+1}, \ldots, t_n) \in P^n$,

$$H_k(t) = 1.$$

It is easy to see that H_k is holomorphic on P^n and satisfies the following conditions: $H_k(0) = 1$, $\operatorname{re}(H_k(t)) \geqslant 0$ for $t \in P^n$. Taking the function H_k as a function of one complex variable t_k (with other variables fixed) we obtain (by *theorem 2*, p. 365 from [4])

$$|H_k(t_1, \ldots, t_n)| \leqslant \frac{1 + |t_k|}{1 - |t_k|} \quad \text{for} \quad t = (t_1, \ldots, t_n) \in P^n.$$

Let $z = (z_1, \ldots, z_n)$ be any point of $E_k^n - \{0\}$. Put $t_i^o = \frac{z_i}{z_k}$ for $i \neq k$, $1 \leqslant i \leqslant n$, and $t_k^o = z_k$. It is obvious that $t_o = (t_1^o, \ldots, t_n^o) \in P^n$ and, since $H_k(t_o) = F_k(z)$, therefore

$$F_k(z) \leqslant \frac{1 + |z_k|}{1 - |z_k|} \tag{2}$$

By the free choise of z, we obtain that inequality (2) takes place for all $z \in E_k^n - \{0\}$. This implies that

$$|h_k(z)| \leqslant |z_k| \frac{1 + |z_k|}{1 - |z_k|} \quad \text{for} \quad z \in E_k^n - \{0\}.$$

Further, observe that $(re^{i\varphi_1}, \ldots, re^{i\varphi_n}) \in E_k^n - \{0\}$ for any $r \in (0, 1)$ and $\varphi_m \in [0, 2\pi]$, $m = 1, \ldots, n$. Hence the inequality

$$|h_k(re^{i\varphi_1}, \ldots, re^{i\varphi_n})| \leqslant r \frac{1 + r}{1 - r}$$

takes place for $r \in (0, 1)$ and $\varphi_m \in [0, 2\pi]$, $m = 1, \ldots, n$. Considering the form of the Bergman-Šilov boundary for the polydisk P^n, we obtain that

$$|h_k(z)| \leqslant \|z\| \frac{1 + \|z\|}{1 - \|z\|} \quad \text{for} \quad z \in P^n$$

From the arbitrariness of k ($1 \leqslant k \leqslant n$) we have

$$\|h(z)\| \leqslant \|z\| \frac{1 + \|z\|}{1 - \|z\|} \quad \text{for} \quad z \in P^n.$$

Lemma 3. The set $M(P^n)$ is closed in $H(P^n)$.

P r o o f. Let $\{h_m\}_{m=1}^{\infty} \subset M(P^n)$ be a sequence converging to $h \in H(P^n)$. Let $h_m = (h_{1m}, \ldots, h_{nm})$ for $m \in N$, From the definition of $M(P^n)$ we have that

$$\operatorname{re} \frac{h_{km}(z)}{z_k} \geqslant 0$$

for $\|z\| = |z_k| > 0$, $k = 1, \ldots, n$ and $m = 1, 2, \ldots$ From the convergence of the sequence $\{h_m\}_{m=1}^{\infty}$ to h we obtain that

$$\operatorname{re} \frac{\tilde{h}_k(z)}{z_k} \geqslant 0$$

for $\|z\| = |z_k| > 0$, where $h = (\tilde{h}_1, \ldots, \tilde{h}_n)$. Since $h(0) = 0$ and $Dh(0) = I$, therefore $h \in M(P^n)$.

Theorem 2. The set $M(P^n)$ is compact in $H(P^n)$.

P r o o f. Let $K \subset P^n$ be a compact set. By *lemma 2*, there exists a number $m_K > 0$ such that $\|h(z)\| \leqslant m_K$ for $z \in K$ and for any $h \in M(P^n)$. Hence, in virtue of the generalized theorem of Montel (see [5], p. 68) and *lemma 3*, we obtain that $M(P^n)$ is compact.

Remark. Directly from the definition of the class $M(P^n)$ it follows that this set is convex, thus (by *theorem 4*, p. 93, from [3]) connected.

Lemma 4. The set $\mathcal{G}_0(P^n)$ is relatively compact in $H(P^n)$.

P r o o f. With the application of *theorem 6* from [6], the proof of this lemma runs similary as that of *theorem 2*.

Now, we shall consider a map $F : M(P^n) \to \mathcal{G}_0(P_n)$ defined in following way. Let $h \in M(P^n)$, then $F(h) = f$ where f is holomorphic on P^n, $f(0) = 0$, $Df(0) = I$ and f fulfils equation (1). The correctness of the definition of F follows immediatly from *theorem 1*.

Theorem 3. The map F is continuous on $M(P^n)$.

P r o o f. Let $\{h_m\}_{m=1}^{\infty} \subset M(P^n)$ be a sequence converging to some function $h \in H(P^n)$. By *lemma 2*, we have that $h \in M(P^n)$. Let $f_m = F(h_m)$ and $f = F(h)$. Suppose that the sequence $\{f_m\}_{m=1}^{\infty}$ does not converge to f. By the relative compactness, we can choose two subsequences $\{f_{k_m}\}_{m=1}^{\infty}$, $\{f_{l_m}\}_{m=1}^{\infty}$ which converge to functions $\tilde{f}$ and $\tilde{\tilde{f}}$, respectively, such that they belong to $H(P^n)$, $\tilde{f}(0) = 0$, $\tilde{\tilde{f}}(0) = 0$, $D\tilde{f}(0) = I$ and $D\tilde{\tilde{f}}(0) = I$. Since the functions f_m for each $m \in N$ fulfil the equation $Df_m(z)h_m(z) = f_m(z)$ for $z \in P^n$, therefore $\tilde{f}$ and $\tilde{\tilde{f}}$ satisfy equation (1). This contradicts the uniqueness of the solution of equation (1) with the conditions $Df(0) = I$ and $f(0) = 0$ (see *theorem 1*).

Theorem 4. The set $\mathcal{G}_0(P^n)$ is compact in $H(P^n)$.

P r o o f. Observe that from the definition of F it follows that $F(M(P^n)) = \mathcal{G}_0(P^n)$. Since, in virtue of *theorem 3*, F is continuous and, by *theorem 2*, the set $M(P^n)$ is compact, therefore, by *theorem 3.17.9.* from [1], $\mathcal{G}_0(P^n)$ is compact in $H(P^n)$.

Theorem 5. The set $\mathcal{G}_0(P^n)$ is connected in $H(P^n)$.

P r o o f. By the continuity of F and the connectedness of $M(P^n)$ (see remark) and *theorem 3.19.7.* from [1], we obtain that the set $\mathcal{G}_0(P^n) = F(M(P^n))$ is connected.

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O PEWNYCH WŁASNOŚCIACH TOPOLOGICZNYCH KLASY ODWZOROWAŃ GWIAŹDZISTYCH I UNORMOWANYCH POLICYLINDRA JEDNOSTKOWEGO W $\mathbb{C}^n$

W pracy tej rozważana jest pewna podklasa klasy odwzorowań jednokrotnych holomorficznych policylindra jednostkowego w $\mathbb{C}^n$. Na początku przedstawiony został pewien warunek konieczny i dostateczny na to, aby odwzorowanie było gwiaździste. Podstawowy rezultat tej pracy to wykazanie, że unormowana klasa odwzorowań gwiaździstych jest zwarta i spójna.