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EXTENSION OF KMENTA'S METHOD FOR THE ESTIMATION OF THE PARAMETERS OF CES PRODUCTION FUNCTION

1. INTRODUCTION

Consider a two-factor CES production function of the form

$$f(K,L) = A\left[bK^{-r} + (1-b)\,L^{-r}\right]^{-\frac{c}{r}}, \tag{1}$$

where K,L denote outlays of capital and labour, respectively. Assuming that the output of production process is a random variable depending on random term ε, the two simplest stochastic models are considered:

$$Y = f(K,L) + \varepsilon \tag{2}$$

or

$$Y = f(K,L)e^{\varepsilon}. \tag{3}$$

The choice of (2) or (3) as the model describing production process determines the method of estimation that can be applied to the estimation of production function parameters.

Out of the well-known and commonly accepted methods of estimation, the Gauss-Newton's and Marquardt's methods are applied to the model (2) and the Kmenta's method to the model (3).

The applicability of Kmenta's method is limited because it was developed under the assumption that substitution elasticity slightly differs from unity. That method was often used in em-

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pirical studies, while no systematic investigations of estimator properties of the parameters obtained using that method were performed.

Section 2 of the present paper shows the results of numerical experiments aiming at an analysis of Kmenta's method accuracy due to interrelationships between the obtained estimates of some parameters and real values of these and other parameters of CES production function.

Developing Kmenta's idea the author obtained (cf. Section 3) an extended model which was the basis for the determination of model parameters (3). The application of the extended model aims at diminishing the systematic errors of the Kmenta's method. However, it causes that the restrictions should be taken into account in the process of estimation.

Section 4 presents the proposal of new estimators of CES function parameters based on the Kmenta's model estimators. It also shows possible determination of mean errors of model parameter estimates (3).

2. KMENTA'S METHOD AND ITS ACCURACY

The method is based on the log-transformation of (3), i.e.,

$$(4) \qquad \ln Y = \begin{cases} \ln A - \frac{c}{r}\ln\,(bK^{-r} + (1-b)L^{-r}) + \varepsilon, & \text{for} \quad r \neq 0, \\ \ln A + cb\ln K + c\,(1-b)\ln L + \varepsilon, & \text{for} \quad r \neq 0. \end{cases}$$

The expression $\ln\,(bK^{-r} + (1-b)L^{-r})$ in (4) is then regarded as a function of variable r. The function $g(r) = \ln(bK^{-r} + (1-b)L^{-r})$ is expanded in the Maclaurin's series taking into account two initial terms. Finally it leads to the estimation of the following model

$$(5) \qquad \ln Y = a_0 + a_1x_1 + a_2x_2 + a_3x_3 + \eta,$$

where

$$a_0 = \ln A, \qquad x_1 = \ln K,$$

$$a_1 = cb, \qquad x_2 = \ln L,$$
$$a_2 = c(1-b), \qquad x_3 = (\ln K - \ln L)^2.$$
$$a_3 = \frac{1}{2} cb(1-b)r,$$

The variable η is the sum of two components: ε and the residual $R_3(\Theta_r)$ out of expansion of the expression $-\frac{g(r)}{r}1$ in series, i.e. - more precisely -

$$\eta = \varepsilon + R_3(\Theta_r).$$

Having estimated parameters a_o, a_1, a_2, a_3 of model (5) the estimators of parameters A, b, c, r of model (4) have been found under the assumption that the relations between estimates of these parameters are identical to the relations between paramaters.

Accuracy of Kmenta's method was analysed by means of an experiment [2]. It consists in that for an established sequence $\{(K_i, L_i)\}$ (consisting of 20 elements) and various combinations of parameters (A, b, c, r) such values of variable Y were determined which satisfy (3) and the assumption that $R^2 = 1$. It means that Y was treated as a deterministic function equal f(K, L).

The following values of parameters were assumed:

A = 0.5, 1.0,

c = 0.5, 1.0, 1.5,

b = 0.9, 0.7, 0.5, 0.3, 0.1,

r = -0.9, -0.5, -0.1, 0.3, 0.7, 1.1, 1.5.

and these gave 210 theoretical combinations of parameter values for (1), and therefore we have obtained 210 artificial samples $\{(Y_i, K_i, L_i)\}$. OLS was applied to each of it to estimate parameters of model (5). Variable η was represented only by $R_3(\Theta_r)$. Estimates $\hat{a}_o$, $\hat{a}_1$, $\hat{a}_2$, $\hat{a}_3$ enabled us to establish estimates $\hat{A}$, $\hat{b}$, $\hat{c}$, $\hat{r}$. According to pre-expectation, the best results were received for $r = -0.1$ and $r = 0.3$. The dependence of estimates $\hat{b}$ and $\hat{r}$ on the real values r and b was observed, when there was no dependence on the values of parameters A and c.

Table 1

Estimates of parameter r depending on values of paramater b

b	Estimates $\hat{r}$						
0.9	-0.283	-0.234	-0.085	-0.434	0.961	1.209	1.357
0.7	-0.373	-0.289	-0.091	0.324	0.612	0.781	0.916
0.5	-0.501	-0.367	-0.098	0.260	0.442	0.552	0.648
0.3	-0.704	-0.492	-0.107	0.216	0.337	0.404	0.462
0.1	-1.107*	-0.745	-0.117	0.184	0.263	0.298	0.323
Real value of r	-0.900	-0.500	-0.100	0.300	0.700	1.100	1.500

*Denotes the parameter estimate exceeding the range of r $(r > -1)$.

Table 2

Estimates of parameter b depending on values of parameter r

r	Estimates $\hat{b}$				
-0.9	0.145	0.286	0.440	0.655	0.858
-0.5	0.126	0.304	0.480	0.706	0.889
-0.1	0.101	0.301	0.500	0.699	0.899
0.3	0.107	0.312	0.506	0.695	0.889
0.7	0.128	0.351	0.539	0.704	0.861
1.1	0.156	0.400	0.581	0.725	0.852
1.5	0.189	0.451	0.623	0.748	0.850
Real value of b	0.100	0.300	0.500	0.700	0.900

The results of studies presented above, although partial, prove the error of Kmenta's method to be considerable. The more r differs from zero, the more significant the error is.

Estimates obtained for parameter A were characterized with a mean error about 3%, when estimates of parameter c showed rather strong stability; errors of estimation were increasing from 0.02% up to 0.07% simultaneously according to the increase of the parameter value. The parameter c is such that it can be treated as estimated nearly without error (for the case of deterministic model).

It is rather obvious that the estimates of model (5) parameters will be inferior to the given above in the case when the variable η will cover both components ε and $R_3(\theta_r)$. Therefore, it is necessary to reduce the influence of the part $R_3(\theta_r)$ of which depends on the accuracy of expansion of a given function in Maclaurin's series. This can be done by means of enlarging the number of terms included explicitly into the model (5).

3. MODIFICATION OF KMENTA'S MODEL

The available works do not present formulae for further (apart from the second) terms of expansion of the function $-\frac{c}{r}\,g(r)$ in power-series. By expanding g(r) in Maclaurin's series up to the fourth term we obtain:

$$g(r) = -(b \ln K + (1-b) \ln L)r + \frac{1}{2} b(1-b)(\ln K - \ln L)^2 r^2 -$$

$$- \frac{1}{6} b(1-b)(1-2b)(\ln K - \ln L)^3 r^3 +$$

$$+ \frac{1}{24} b(1-b)(6b^2 - 6b + 1)(\ln K - \ln L)^2 r^4 + R_5(\theta_r),$$

where $\theta \in (0; 1)$.

Using standard algebraic operations we obtain:

$$- \frac{c}{r} g(r) = c \ln L + cb \ln \frac{K}{L} - \frac{1}{2} cb(1-b) r \ln^2 \frac{K}{L} +$$

$$+ \frac{1}{3} cb(1-b)\left(\frac{1}{2} - b\right) r^2 \ln^3 \frac{K}{L} +$$

$$- \frac{1}{4} cb(1-b)\left(b - \frac{1}{2} + \frac{1}{2\sqrt{3}}\right)\left(b - \frac{1}{2} - \frac{1}{2\sqrt{3}}\right) r^3 \ln^4 \frac{K}{L} +$$

$$+ \bar{R}_5(\theta r),$$

$$\bar{R}_5(\theta r) = - \frac{c}{r} R_5(\theta r).$$

Finally the modified Kmenta's model is of the form:

$$(6)\qquad \ln Y = a_0 + a_1 \ln K + a_2 \ln L + a_3x^2 + a_4x^3 + a_5x^4 + \eta$$

where

$$(7)\qquad \begin{aligned} a_0 &= \ln A, \\ a_1 &= cb, \\ a_2 &= c(1-b), \\ a_3 &= \frac{1}{2} cb(1-b)r, \\ a_4 &= \frac{1}{6} cb(1-b)(1-2b)r^2, \\ a_5 &= -\frac{1}{24} cb(1-b)(6b^2 - 6b + 1)\, r^3, \end{aligned}$$

and

$$x = \ln K - \ln L.$$

System (7) consists of six equations containing four unknown parameters A, b, c, r. Solving the four initial equations of (7) we get:

$$(8)\qquad \begin{aligned} A &= \exp(a_0), \\ b &= \frac{a_1}{a_1 + a_2}, \\ c &= a_1 + a_2, \\ r &= -2\left(\frac{a_3}{a_1} + \frac{a_3}{a_2}\right), \end{aligned}$$

and simultaneously parameters a_1, a_2, a_3, a_4, a_5 must hold the following identities:

$$(9)\qquad \begin{cases} 3\, a_1 a_2 a_4 = 2\, a_3^2(a_2 - a_1) \\ 6\, a_1 a_2 (a_1 + a_2) a_5 = a_3^2(4\, a_1 a_2 - a_1^2 - a_2^2). \end{cases}$$

In order to apply the OLS to estimate parameters of the model (6) we should include the constraints (9). It is worthwhile to pay attention to a specific construction of model (6), i.e., to variables x^2, x^3, x^4. As successive powers of the same variable they are - in principle - likely to be highly correlated. It causes the increase of numerical errors of the estimator values.

4. REMARKS ON THE DISTRIBUTIONS OF ESTIMATORS FOR THE KMENTA'S MODEL

Assuming that the random term ε is normally distributed and that the influence of $\bar{R}_5(\Theta r)$ is practically insignificant, we expect the parameter estimates $\hat{a}_o$, $\hat{a}_1$, $\hat{a}_2$, $\hat{a}_3$, $\hat{a}_4$, $\hat{a}_5$ to be also normally distributed. Accepting conventional (and convenient) assumption that estimates of parameters A, b, c, r are given by a relation similar to (8), we can find moments of distribution as well as distributions of estimates $\hat{A}$ and $\hat{c}$. Identical problem cannot be solved for $\hat{b}$ and $\hat{r}$, as in this case the knowledge of distributions of random variables $\frac{1}{\hat{a}_1}$, $\frac{1}{\hat{a}_2}$, $\frac{1}{\hat{a}_1 + \hat{a}_2}$ is necessary. In the literature on the subject approximate formulae can be found for the expected value and variance of random variable $Z = h(U)$ when the moments $E(U)$ and $D^2(U)$ are known. For the case $Z = 1/U$ we have

$$(10)\qquad \begin{aligned} E(1/U) &\cong 1/\mu, \\ D^2(1/U) &\cong \sigma^2/\mu^4, \end{aligned}$$

where: $\mu = E(U) \neq 0$ and $D^2(U) = \sigma^2$. But it is hard to say what formulae (10) approximate because for $U : N(\mu, \sigma)$ the integral

$$\int_{-\infty}^{\infty} \frac{1}{u} f(u)du$$

is divergent (where $f(u)$ is the density function of random variable U).

It has been shown [4] that moments of random variable $1/U$ can be found if it is assumed that U has a left-side truncated normal distribution, i.e. if density function $f_a(u)$ is of the form

$$f_a(u) = \begin{cases} 0 & \text{for } u < a, \\ \dfrac{f(u)}{1 - \Phi\left(\frac{a-\mu}{\sigma}\right)} & \text{for } u \geqslant a, \end{cases} \tag{11}$$

where: $\Phi(u)$ is the distribution function of a random variable having normal distribution $N(0,1)$.

The tables of values for $E(1/X_o)$ and $E(1/X_o^2)$ are given for $X_o : N_a(\mu, 1)$ and for the various levels of truncation points a and values of μ (see [4]). Proposition is also made to accept truncation level a_{KR} for the case when no additional information outside the sample existed. Values of a_{KR} were found as a minimum of the integrated function for the integral determining $E(1/X_o)$. This minimum exists (the condition for the random variable U_o to have truncated normal distribution $U_o : N_a(\mu_u, \sigma_u)$ is equivalent to the relation $\mu_u > 2\sigma_u$) for $\mu > 2$.

Let the estimates $\hat{a}_1$, $\hat{a}_2$, $\hat{a}_1 + \hat{a}_2$ have truncated normal distributions, i.e.,

$$\hat{a}_1 : N_{a_I}(\mu_1, \sigma_1), \quad \hat{a}_2 : N_{a_{II}}(\mu_2, \sigma_2),$$

$$\hat{c} = (\hat{a}_1 + \hat{a}_2) : N_{a_{III}}\left(\mu_1 + \mu_2, \sqrt{\sigma_1^2 + 2\text{cov}(\hat{a}_1\hat{a}_2) + \sigma_2^2}\right),$$

where a_I, a_{II}, a_{III} are the levels of truncation for the respective estimates.

Then we get:

$$E(1/\hat{a}_i) = \frac{1}{\sigma_2} E(1/X_o),$$

(12)

$$D^2(1/\hat{a}_i) = \frac{1}{\sigma_i^2} (E(1/X_o^2) - (E(1/X_o))^2),$$

where:

$$X_o : N_{a^*}\left(\frac{\mu_i}{\sigma_i}, 1\right) \quad \text{for} \quad i = 1,2$$

and

$$E(1/\hat{c}) = \frac{1}{\sigma_3} E(1/X_o),$$

(13) $$D^2(1/\hat{c}) = \frac{1}{\sigma_3^2} \left[E(1/X_o^2) - (E(1/X_o))^2\right]$$

for

$$X_o : N_{a^*}\left(\frac{\mu_1 + \mu_2}{\sigma_3}, 1\right),$$

(13) $$\sigma_3 = \sqrt{\sigma_1^2 + 2\mathrm{cov}(\hat{a}_1, \hat{a}_2) + \sigma_2^2},$$

where

$$a^* = \min\left\{\frac{a_I}{\sigma_1}, \frac{a_{II}}{\sigma_2}, \frac{a_{III}}{\sigma_3}\right\}.$$

Making consequent use of the results obtained by P i e t c o l d, T o m a s z e w i c z, Ż ó ł t o w s k a [3] we obtain:

$$E(\hat{b}) = E(\hat{a}_1/\hat{c}) = QE(1/X_o) + R,$$

(14) $$D^2(\hat{b}) = D^2(\hat{a}_1/\hat{c}) = SE(1/X_o^2) + Q^2(E(1/X_o^2) - (E(1/X_o))^2),$$

where:

$$R = \varrho(\hat{a}_1, \hat{c}) \frac{\sigma_3}{\sigma_1}$$

(15) $$Q = \frac{\sigma_3}{\sigma_1}\left(\frac{\mu \hat{a}_1}{\sigma_1} - \rho(\hat{a}_1,\hat{c})\frac{\mu\hat{c}}{\sigma_3}\right), \quad S = \frac{\sigma_1^2}{\sigma_3^2}(1 - \rho^2(\hat{a}_1,\hat{c})).$$

We do not know the values of moments for estimates $\hat{a}_1$ and $\hat{c}$ but we have their estimates obtained from the sample (these estimates are obtained as a result of the estimation of model (6)). Making use of relation (15), equation (14) and also tables of functions $E(1/X_o)$ and $E(1/X_o^2)$ presented earlier [4] the estimates of parameters $E(\hat{b})$ and $D^2(\hat{b})$ can be obtained.

The following formulae are evaluated in an analogical manner giving

$$E(\hat{r}) = -2(E(\hat{a}_3/\hat{a}_1) + E(\hat{a}_3/\hat{a}_2))$$

and

$$D^2(\hat{r}) = 4(D^2(\hat{a}_3/\hat{a}_1) + 2\,\mathrm{cov}(\hat{a}_3/\hat{a}_1,\ \hat{a}_3/\hat{a}_2) + D^2(\hat{a}_3/\hat{a}_2)).$$

To determine $D^2(\hat{r})$ it is necessary to calculate $\mathrm{cov}(\hat{a}_3/\hat{a}_1, \hat{a}_3/\hat{a}_2)$ and more precisely,

$$E\left(\frac{\hat{a}_3^{\,2}}{\hat{a}_1, \hat{a}_2}\right).$$

In the first step the linear transformation

$$\begin{bmatrix} \hat{a}_1 \\ \hat{a}_2 \\ \hat{a}_3 \end{bmatrix} = A \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$$

should be determined. The transformation brings multidimensional random variable

$$\begin{bmatrix} \hat{a}_1 \\ \hat{a}_2 \\ \hat{a}_3 \end{bmatrix} \text{ with the distribution } N\left(\begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{bmatrix}, \begin{bmatrix} \sigma_1^2 & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_2^2 & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_3^2 \end{bmatrix}\right)$$

into the multidimensional variable

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \text{ with the distribution } N\left(\begin{bmatrix} \mu_x \\ \mu_y \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right).$$

We obtain:

$$\hat{a}_1 = aX,$$

$$\hat{a}_2 = bX + cY,$$

$$\hat{a}_3 = dX + eY + fZ + g,$$

where:

$$a = \sigma_1, \quad b = r_{12}\sigma_2, \quad c = \sigma_2\sqrt{1 - r_{12}^2}, \quad d = r_{13}\sigma_3,$$

$$e = \sigma_3 \quad (r_{23} - r_{12}r_{13})/(1 - r_{12}^2), \quad f = \sigma_3\sqrt{\left(1 - r_{23}^2\right)\left(1 - r_{23.1}^2\right)}$$

$$g = \mu_3 - d\mu_x - e\mu_y.$$

Expected value $\frac{\hat{a}_3^2}{\hat{a}_1 \hat{a}_2}$ was found as:

$$E\left(\frac{\hat{a}_3^2}{\hat{a}_1 \hat{a}_2}\right) = \beta_1 + \beta_2 E\left(\frac{1}{X}\right) + \beta_3 E\left(\frac{1}{bX + cY}\right) + \beta_4 E\left(\frac{1}{X(bX + cY)}\right) +$$

$$+ \beta_5 E\left(\frac{X}{bX + cY}\right),$$

where:

$$\beta_1 = \frac{e}{ca}\left(2d + \frac{be}{c}\right),$$

$$\beta_2 = c\,\mu_y + \frac{2e}{ca}\left(d - \frac{be}{c}\right),$$

$$\beta_3 = \frac{2e}{ac}\left(d - \frac{be}{c}\right),$$

$$\beta_4 = \frac{f^2}{a} + \frac{e^2}{c^2 a},$$

$$\beta_5 = \frac{(d - ab)^2}{a}.$$

To find $D^2(\hat{r})$ it is thus necessary to know $E\left(\frac{1}{X(bX + cY)}\right)$ and therefore to determine the moment of reciprocal random variable which is the product of two variables having truncated normal distribution. This problem has not been solved yet.

5. FINAL REMARKS

The author considers a possibility of determining estimators of CES production function parameters with a multiplicate random term which are better than the estimators obtained using the Kmenta's method. Two proposals have been given. The first one concerns the modification of Kmenta's approach by extending the number of elements in a particular model. The other one proposes new estimators based on the moments of products of random variables; this makes it possible to determine mean errors of estimate parameters of the CES production function which has not been done so far.

However, the application of the results presented in this paper to empirical investigations requires an efficient method of estimation of linear model parameters to be developed. In such a linear model there are subsequent natural powers of the same explanatory variable and parameters fulfil a two non-linear equation system. In order to determine efficiently the second central moment for the estimator of a substitution parameter r it is necessary to determine (e.g. by constructing special tables) the expected value of inverse of the product of two dependent random variables with respective truncated distributions.

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ROZSZERZENIE METODY KMENTY DO ESTYMACJI PARAMETRÓW FUNKCJI PRODUKCJI TYPU CES

Artykuł zawiera analizę wybranych własności metody Kmenty estymacji parametrów funkcji produkcji typu CES, a w szczególności:

1) opis wyników estymatorów, które rzucają nowe światło na dokładność i efektywność tej metody oraz ukazują wzajemne relacje między otrzymywanymi za jej pomocą ocenami parametrów funkcji produkcji typu CES;

2) propozycję (w celu porównania dokładności) rozwinięcia metody Kmenty, która polega na:

a - uzupełnieniu modelu Kmenty o dwa dodatkowe składniki przy jednoczesnym wskazaniu warunków pobocznych, jakie powinny spełniać parametry rozszerzonego modelu,

b - znalezieniu nowych estymatorów parametrów funkcji CES, (wykorzystując ogólną ideę metody Kmenty), w postaci momentów ilorazów estymatorów parametrów modeli Kmenty oraz wskazaniu możliwości wyznaczania średnich błędów dla estymatorów.