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REMARK ON SOME THEOREM OF ZAJÍČEK

For a typical continuous function f on $[0,1]$, f has an I-essential derived number at each point $x \in (0,1)$.

Zajíček proved [3] that, for a typical continuous real-valued function f and each $x \in (0,1)$, there exists $y \in \bar{\mathfrak{R}}$ which is an essential derived number of f at x. In this paper we shall prove that this theorem remains true if we replace the notion of an essential derived number of f at x by an analogous notion for the Baire category.

Let $\mathcal{C}$ denote the set of continuous real valued functions defined on $[0,1]$ furnished with the metric of uniform convergence. When we say a typical $f \in \mathcal{C}$ has a certain property $\mathcal{P}$, we shall mean that the set of $f \in \mathcal{C}$ with this property is residual in $\mathcal{C}$.

The notation used throughout this paper is standard. In particular, $\mathfrak{R}$ stands for the set of real numbers, $\bar{\mathfrak{R}} = \mathfrak{R} \cup \{-\infty, \infty\}$, I for the σ-ideal of sets of the first category, $\|f\|$ for the norm in $\mathcal{C}$, $B(f,r)$ for the open ball in $\mathcal{C}$ with centre f and radius r and χ_A for the characteristic function of a set A.

Definition 1. ([1]) We say that $x_0 \in \mathfrak{R}$ is an upper I-density point of a set A having the Baire property if and only if there exists an increasing sequence of real numbers $\{t_n\}_{n \in N}$ tending to infinity, such that

$$\chi_{t_n(E-x_0)\cap(-1,1)} \longrightarrow 1 \textit{ withrespectto } I \text{ as } n \to \infty\text{on } (-1,1)$$

(see [2] for the definition of the convergence with respect to I). We shall use the notation $\bar{d}_I(E, x_0) = 1$.

Observe that x_0 is an upper I-density point of a set A if and only if 0 is an upper I-density point of $A - x_0 = \{x - x_0 : x \in A\}$.

It is easy to see that $\bar{d}_I(E, x_0) = 1$ if and only if there exists an increasing sequence of real numbers $\{t_n\}_{n\in N}$ tending to infinity, such that

$$\lim_{n\to\infty} \chi_{t_n(E-x_0)\cap(-1,1)}(x) = 1$$

I-a.e. on (-1,1).

Definition 2. We say that y is an I-essential derived number of f at x if there exists a set $E \subset \Re$ having the Baire property, such that $\bar{d}_I(E, x) = 1$ and $\lim_{t\to x, t\in E} \frac{f(t)-f(x)}{t-x} = y$.

Theorem. *For a typical $f \in \mathcal{C}$ and each $x \in (0,1)$, there exists $y \in \bar{\Re}$ which is an I-essential derived number of f at x.*

Proof. Let $\{P_k\}_{k\in N}$ be a sequence of polynomials which is dense in $\mathcal{C}$. For each $k \in N$, put $M_k = \|P_k''\| = \sup_{x\in[0,1]} |P_k''(x)|$ and choose δ_k such that $0 < \delta_k < (kM_k)^{-1}$ and $\delta_k \searrow 0$ as $k \to \infty$. Let

$$G = \bigcap_{m=1}^{\infty} \bigcup_{k=m}^{\infty} B\left(P_k, \frac{\delta_k}{4k^2}\right) = \limsup_k B\left(P_k, \frac{\delta_k}{4k^2}\right).$$

For each $m \in N$, the set $\bigcup_{k=m}^{\infty} B(P_k, \frac{\delta_k}{4k^2})$ is open and dense, so G is a dense G_δ-subset of $\mathcal{C}$. Hence G is residual in $\mathcal{C}$. Choose an arbitrary $f \in G$. It is sufficient to prove that, for each $x \in (0,1)$, there exists an I-essential derived number of f at x. Fix $x_0 \in (0,1)$. Since $f \in G$, we can choose an increasing sequence of positive integers $\{k_n\}_{n\in N}$ such that $f \in B(P_{k_n}, \delta_{k_n} \cdot (4k_n^2)^{-1})$ for each $n \in N$. Let $h_n = \delta_{k_n}$, $A_n = P'_{k_n}(x_0)$, $z_n = (k_n)^{-1}$. Since $\delta_n \searrow 0$ as $n \to \infty$, we have $h_n \searrow 0$ for $n \to \infty$. For n large enough, we get $(x_0 - h_n, x_0 + h_n) \subset (0,1)$. We shall show that, for such an n and for $x \in (x_0 - h_n, x_0 - h_n(k_n)^{-1}) \cup (x_0 + h_n(k_n)^{-1}, x_0 + h_n)$, we have

$$\left| \frac{f(x) - f(x_0)}{x - x_0} - A_n \right| < z_n.$$

We can assume that $x \in (x_0 + \frac{h_n}{k_n}, x_0 + h_n)$ (the other case is analogous). We have

$$\begin{aligned} &\left| \frac{f(x) - f(x_0)}{x - x_0} - \frac{P_{k_n}(x) - P_{k_n}(x_0)}{x - x_0} \right| \\ (1) \qquad &\leq \frac{|f(x) - P_{k_n}(x)| + |f(x_0) - P_{k_n}(x_0)|}{|x - x_0|} \\ &< \frac{2\delta_{k_n}(4k_n^2)^{-1}}{\delta_{k_n}(k_n)^{-1}} = \frac{1}{2k_n}. \end{aligned}$$

By the Taylor formula, for some $\xi \in (0,1)$,

$$\begin{aligned} &\left| \frac{P_{k_n}(x) - P_{k_n}(x_0)}{x - x_0} - P'_{k_n}(x_0) \right| = \left| \frac{1}{2} P''_{k_n}(\xi)(x - x_0) \right| \\ (2) \qquad &< \frac{1}{2} M_{k_n} \delta_{k_n} < \frac{1}{2} M_{k_n} \cdot \frac{1}{k_n M_{k_n}} \\ &= \frac{1}{2k_n}. \end{aligned}$$

From (1) and (2) we obtain

$$\left| \frac{f(x) - f(x_0)}{x - x_0} - P'_{k_n}(x_0) \right| < \frac{1}{k_n},$$

hence

$$(3) \qquad \left| \frac{f(x) - f(x_0)}{x - x_0} - A_n \right| < z_n$$

for $x \in (x_0 - h_n, x_0 - h_n(k_n)^{-1}) \cup (x_0 + h_n(k_n)^{-1}, x_0 + h_n)$.

Denote by $y \in \bar{\mathfrak{R}}$ a cluster point of a sequence $\{A_n\}_{n\in N}$. Then there exists a subsequence $\{n_p\}_{p\in N}$ of the sequence of positive integers, such that $A_{n_p} \to y$ as $p \to \infty$. Define

$$E = \bigcup_{p=1}^{\infty} \left[\left(x_0 - h_{n_p}, x_0 - \frac{h_{n_p}}{k_{n_p}} \right) \cup \left(x_0 + \frac{h_{n_p}}{k_{n_p}}, x_0 + h_{n_p} \right) \right].$$

We shall show that

$$\bar{d}_I(E, x_0) = 1 \text{ and } \lim_{\substack{x \to \infty \\ x \in E}} \frac{f(x) - f(x_0)}{x - x_0} = y.$$

According to the above remarks, it is sufficient to show that there exists a sequence of real numbers $\{t_n\}_{n\in N}$ tending to infinity, such that

$$\lim_{n\to\infty} \chi_{t_n(E-x_0)\cap(-1,1)}(x) = \chi_{(-1,1)}(x)$$

I-a.e. on $(-1,1)$. Let $t_p = (h_{n_p})^{-1}$. Then

$$t_p \cdot (E - x_0) \supset t_p \left[\left(-h_{n_p}, -\frac{h_{n_p}}{k_{n_p}} \right) \cup \left(\frac{h_{n_p}}{k_{n_p}}, h_{n_p} \right) \right]$$
$$= \left(-1, -\frac{1}{k_{n_p}} \right) \cup \left(\frac{1}{k_{n_p}}, 1 \right).$$

Since $h_{n_p} \searrow 0$ for $p \to \infty$ and $k_n \nearrow \infty$ for $n \to \infty$, we have $t_p \nearrow \infty$ as $p \to \infty$ and $1/k_{n_l} \searrow 0$ as $l \to \infty$. Hence, for $x \in (0,1)$, $x \neq 0$, there exists p_0 such that, for each $p \geq p_0$,

$$x \in \left(-1, -\frac{1}{k_{n_p}} \right) \cup \left(\frac{1}{k_{n_p}}, 1 \right).$$

Then

$$x \in t_p \cdot \left[\left(-h_{n_p}, -\frac{h_{n_p}}{k_{n_p}} \right), \left(\frac{h_{n_p}}{k_{n_p}}, h_{n_p} \right) \right] \subset t_p \cdot (E - x_0),$$

so, for $p \geq p_0$, we have

$$\chi_{t_p(E-x_0)\cap(-1,1)}(x) = 1.$$

Therefore $\bar{d}_I(E, x_0) = 1$.

We only need to show that

$$\lim_{\substack{x \to x_0 \\ x \in E}} \frac{f(x) - f(x_0)}{x - x_0} = y.$$

Let $\varepsilon > 0$. There exists p_0 such that $1/k_{n_p} < \varepsilon/2$ for $p \geq p_0$. Then, for each $x \in E$ such that $|x - x_0| < h_{n_{p_0}}$, we have

$$x \in \bigcup_{p=p_0}^{\infty} \left[\left(x_0 - h_{n_p}, x_0 - \frac{h_{n_p}}{k_{n_p}} \right) \cup \left(x_0 + \frac{h_{n_p}}{k_{n_p}}, x_0 + h_{n_p} \right) \right].$$

From (3) it follows that there exists $p \geq p_0$ such that

$$\left| \frac{f(x) - f(x_0)}{x - x_0} - A_{n_p} \right| < z_{n_p} = \frac{1}{k_{n_p}} < \frac{\varepsilon}{2}.$$

Since $A_{n_p} \to y$ as $p \to \infty$, there exists p_1 such that, for $p \geq p_1$, we get

$$|A_{n_p} - y| < \frac{\varepsilon}{2}.$$

Let $p_2 = max(p_0, p_1)$ and $\delta = h_{n_{p_2}}$. According to the above remarks, for $x \in E$, if $|x - x_0| < \delta$, then we have

$$\left| \frac{f(x) - f(x_0)}{x - x_0} - y \right| < \varepsilon.$$

This completes the proof.

References

[1] T. Filipczak, *On some abstract density topologies*, Real Analysis Exchange **14** (1988-89), 140–166.

[2] W. Poreda, E. Wagner-Bojakowska and W Wilczyński, *A category analogue of the density topology*, Fund. Math. **125** (1985), 167–173.

[3] L. Zajíček, *On essential derived numbers of typical continuous functions*, Tatra Mountains Math. Publ. **2** (1993), 123–125.

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UWAGA NA TEMAT PEWNEGO TWIERDZENIA L. ZAÍČKA

Zajíček ([3]) udowodnił, że typowa funkcja rzeczywista ma w każdym punkcie $x \in (0,1)$ istotną liczbę pochodną. W pracy tej dowodzimy, że twierdzenie to pozostaje prawdziwe, jeśli zastąpimy pojęcie istotnej liczby pochodnej przez analogiczne pojęcie dla kategorii Baire'a.

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