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UPPER AND LOWER LIMITS WITH RESPECT TO THE $\mathfrak{m}$–IDEAL

The paper deals with the convergence of a transfinite sequence of functions, with respect to $\mathfrak{M}$-ideal I. It is assumed that all functions are S-measurable, where S is a $\mathfrak{M}$-algebra. The main result says that such a sequence converges with respect to I if and only if its upper and lower limits (also with respect to I). coincide. The role of S-measurability of functions is discussed. The results are similar to that of E. Wagner [7] for ordinary sequences.

The notion of upper and lower limits in measure of a sequence of measurable functions were studied by C. Goffman and D. Waterman in [1].

Similar results for category analogues i.e. upper and lower limits in category of a sequence of functions having the Baire property were obtained by E. Wagner in [7] and further generalized in [8] for sequences of S-measurable functions, where S is any σ-field of subsets of a set X. The only assumption on S is that quotient algebra S/I fulfils countable chain condition; $I \subset S$ is here a proper σ-ideal. The equivalence of three different definitions of upper and lower limits with respect to the σ-ideal i.e. in the sense of Menchoff, in the sense of Goffman and Waterman and in the sense of Wagner was shown by W. Wilczyński in [9].

The aim of the paper is to examine similar problems as in [7] replacing usual sequences by transfinite ones.

In [10] one can find four equivalent definitions of convergence with respect to the ideal I (S/I is assumed to satisfy countable chain condition here) for usual sequences of S-measurable functions. Two of them are due to E. Wagner and next two to T. Świątkowski. We choose one of those given by E. Wagner to reformulate it to the case of transfinite sequences.

Let X be an arbitrary non-empty set, S – a fixed $\mathfrak{m}$-field of subsets of X and $I \subset S$ - a proper $\mathfrak{m}$-ideal in S. Identifying the sets $A, B \in S$ if and only if $A \triangle B \in I$ we obtain a quotient Boolean algebra S/I. The class including a set A will be denoted by $[A]$. For classes $[A]$, $[B] \in S/I$ the denotation $[A] \subset [B]$ means that $A_1 \setminus B_1 \in I$ for every $A_1 \in [A]$ and $B_1 \in [B]$. We assume here additionaly that S/I satisfies $\mathfrak{m}$-chain condition.

A real function defined on X is a null function if and only if it is equal to zero I-a.e., two functions f and g which are S-measurable are called equivalent if and only if $f-g$ is a null function. For equivalence classes $[f]$ and $[g]$ the denotation $[f] \leq [g]$ means that $f_1(x) \leq g_1(x)$ I-a.e. on X where $f_1 \in [f]$ and $g_1 \in [g]$.

For $\mathfrak{m}$-sequence of reals $\{x_\alpha\}_{\alpha<\mathfrak{m}}$ we use denotation:

$$\lim_\alpha \sup x_\alpha \text{ for } \inf_{\alpha<\mathfrak{m}} \sup_{\beta\geq\alpha} x_\beta$$
$$\text{and } \lim_\alpha \inf x_\alpha \text{ for } \sup_{\alpha<\mathfrak{m}} \inf_{\beta\geq\alpha} x_\beta.$$

Lemma 1. *The Boolean algebra S/I is a complete lattice if and only if the family $\mathfrak{M}$ of equivalence classes of S measurable real functions on X is a complete lattice.*

Proof. One can follow exactly the proof of Lemma 1 in [7].

Lemma 2. *S/I is a complete lattice.*

Proof. As S is $\mathfrak{m}$-complete we have S/I to be $\mathfrak{m}$-complete by Theorem 21.1 in [5]. S/I satisfies also the $\mathfrak{m}$-chain condition hence by Theorem 20.5 from [5] it is a complete lattice.

Corollary 1. *$\mathfrak{M}$ is a complete lattice.*

Definition 1. We shall say that the sequence $\{f_\alpha\}_{\alpha<\mathfrak{m}}$ of S-measurable functions defined on X converges with respect to I (in the sense of Wagner) to the S-measurable function f defined on X if and only if both of the following conditions are fulfilled:

1) for every subsequence $\{f_{\alpha_\nu}\}_{\nu<\mathfrak{m}}$ of $\{f_\alpha\}_{\alpha<\mathfrak{m}}$, for every set $A \in S \setminus I$ and for every natural number k there exists a subsequence $\{f_{\alpha_{\nu_\gamma}}\}_{\gamma<\mathfrak{m}}$ of $\{f_{\alpha_\nu}\}_{\nu<\mathfrak{m}}$ such that $\lim_\gamma \sup f_{\alpha_{\nu_\gamma}}(x) < f(x) + \frac{1}{k}$ on the set $A' \subset A$ such that $A' \in S \setminus I$.
2) for every subsequence $\{f_{\alpha_\nu}\}_{\nu<\mathfrak{m}}$ of $\{f_\alpha\}_{\alpha<\mathfrak{m}}$, for every set $A \in S \setminus I$ and for every natural number k there exists a subsequence $\{f_{\alpha_{\nu_\gamma}}\}_{\gamma<\mathfrak{m}}$ of $\{f_{\alpha_\nu}\}_{\nu<\mathfrak{m}}$ such that $\lim_\gamma \inf f_{\alpha_{\nu_\gamma}}(x) > f(x) - \frac{1}{k}$ on the set $A" \subset A$ such that $A'' \in S \setminus I$.

If functions $\{f_\alpha\}_{\alpha<\mathfrak{m}}$ in the above definition are not S-measurable we say that the sequence $\{f_\alpha\}_{\alpha<\mathfrak{m}}$ of functions defined on X converges in general with respect to I to the function f defined on X.

We shall now define an equivalence relation for $\mathfrak{m}$-sequences of S-measurable functions. Let $\{f_\alpha\}_{\alpha<\mathfrak{m}}$ be equivalent to $\{g_\alpha\}_{\alpha<\mathfrak{m}}$ if and only if $\{f_\alpha - g_\alpha\}_{\alpha<\mathfrak{m}}$ converges with respect to I to zero. Let $\{f_\alpha\}_{\alpha<\mathfrak{m}}$ be a sequence of S-measurable functions and let $\mathcal{F}$ be the equivalence class including $\{f_\alpha\}_{\alpha<\mathfrak{m}}$.

Definition 2. We shall say that $U \in \mathfrak{M}$ ($L \in \mathfrak{M}$) is upper (lower) limit of a sequence $\{f_\alpha\}_{\alpha<\mathfrak{m}}$ wit respect to I (in the sense of Wagner) if and only if

$$U = \inf[\lim_\alpha \sup g_\alpha : \{g_\alpha\}_{\alpha<\mathfrak{m}} \in \mathcal{F}]$$

$$(L = \sup[\lim_\alpha \inf g_\alpha : \{g_\alpha\}_{\alpha<\mathfrak{m}} \in \mathcal{F}])$$

The existence of U, L follows from Corollary 1 and from the fact that for every sequence $\{g_\alpha\}_{\alpha<\mathfrak{m}}$ of S-measurable functions the functions $\lim_\alpha \sup g_\alpha$ and $\lim_\alpha \inf g_\alpha$ are also S-measurable (proof of the fact is similar to proof of Theorem 10.2vi in [6]).

Lemma 3. *If U and L are upper and lower limits with respect to I of a sequence $\{f_\alpha\}_{\alpha<\mathfrak{m}}$, then $L \leq U$.*

Proof. Suppose that for some $u \in U$ and $l \in L$ we have $u(x) < l(x)$ on a set $B \notin I$. There exists a sequence $\{h_\alpha\}_{\alpha<\mathfrak{m}} \in \mathcal{F}$ such that

$\lim_\alpha \sup h_\alpha(x) < l(x)$ on the set $C \notin I$ and there exists a sequence $\{g_\alpha\}_{\alpha<\mathfrak{m}} \in \mathcal{F}$ such that $\lim_\alpha \sup h_\alpha(x) < \lim_\alpha \inf g_\alpha(x)$ on the set $D \notin I$. We have

$$D = \bigcup_{\substack{w\in W\\ w>0}} \{x : \limsup_\alpha h_\alpha(x) + w < \liminf_\alpha g_\alpha(x)\}$$

where W is the set of all rational numbers, and so there exists a number $w_0 > 0$ such that the set $\{x : \lim_\alpha \sup h_\alpha(x) + w_0 < \lim_\alpha \inf g_\alpha(x)\}$ does not belong to I.

Obviously

$$\begin{aligned}
&\{x : \limsup_\alpha h_\alpha(x) + w_0 < \liminf_\alpha g_\alpha(x)\}\\
&= \{x : \inf_{\alpha<\mathfrak{m}} \sup_{\beta\geq\alpha} h_\beta(x) + w_0 < \sup_{\gamma<\mathfrak{m}} \inf_{\delta\geq\gamma} (x)\}\\
&= \bigcup_{\alpha<\mathfrak{m}} \bigcup_{\gamma<\mathfrak{m}} \{x : \sup_{\beta\geq\alpha} h_\beta(x) + w_0 < \inf_{\delta\geq\gamma} g_\delta(x)\}\\
&= \bigcup_{\xi<\mathfrak{m}} \{x : \sup_{\beta\geq\xi} h_\beta(x) + w_0 < \inf_{\delta\geq\xi} g_\delta(x)\}\\
&= \bigcup_{\alpha<\mathfrak{m}} \{x : \sup_{\beta\geq\alpha} h_\beta(x) + w_0 < \inf_{\beta\geq\alpha} g_\beta(x)\},
\end{aligned}$$

hence there exists ordinal number α_0 such that the set

$$E = \{x : \sup_{\beta\geq\alpha_0} h_\beta(x) + w_0 < \inf_{\beta\geq\alpha_0} g_\beta(x)\}$$

does not belong to I. Hence for every $x \in E$ and for every $\alpha > \alpha_0$ we have $g_\alpha(x) - h_\alpha(x) > w_0 > 0$ and so a sequence $\{g_\alpha\}_{\alpha<\mathfrak{m}}$ is not equivalent to $\{h_\alpha\}_{\alpha<\mathfrak{m}}$ – a contradiction.

Theorem 1. *Let f and $\{f_\alpha\}_{\alpha<\mathfrak{m}}$, be S-measurable functions. A sequence $\{f_\alpha\}_{\alpha<\mathfrak{m}}$ converges with respect to I to a function f if and only if $U = L$, and then $f = U = L$.*

Proof. Suppose that $\{f_\alpha\}_{\alpha<\mathfrak{m}}$ converges to f with respect to I. Then a sequence $\{f_\alpha\}_{\alpha<\mathfrak{m}}$ is equivalent to a sequence $\{g_\alpha\}_{\alpha<\mathfrak{m}}$, where

$g_\alpha = f$ for $\alpha < \mathfrak{m}$. Hence $U \le f$, because $\lim_\alpha \sup g_\alpha = f$. Similarly one can prove that $L \ge f$. So $f \le L \le U \le f$ and $U = L$.

Suppose now that $U = L$. Without loss of generality we can suppose that $U \equiv 0$. Assume that $\{f_\alpha\}_{\alpha<\mathfrak{m}}$ does not converge to zero with respect to I and the first condition of Definition 1 is not fulfilled (in the remaining case the proof is similar).

Let $\{g_\alpha\}_{\alpha<\mathfrak{m}}$ be a sequence equivalent to $\{f_\alpha\}_{\alpha<\mathfrak{m}}$. Then $g_\alpha = f_\alpha + h_\alpha$, $\alpha < \mathfrak{m}$, where $\{h_\alpha\}_{\alpha<\mathfrak{m}}$ is a sequence of S -measurable functions converging to zero with respect to I. A subsequence $\{f_{\alpha_\beta}\}_{\beta<\mathfrak{m}}$, set $A_0 \in S - I$ and natural number k_0 are such that for every subsequence $\{f_{\alpha_{\beta_\gamma}}\}_{\gamma<\mathfrak{m}}$ of $\{f_{\alpha_\beta}\}_{\beta<ff}$ we have $\lim_\gamma \sup f_{\alpha_{\beta_\gamma}}(x) \ge \frac{1}{k_0}$ (I-a.e. on A_0).

For the subsequence $\{h_{\alpha_\beta}\}_{\beta<\mathfrak{m}}$ of $\{h_\alpha\}_{\alpha<\mathfrak{m}}$, set $A_0 \in S\setminus I$ and natural number $l_0 > k_0$ there exists a subsequence $\{h_{\alpha_{\beta_\gamma}}\}_{\gamma<\mathfrak{m}}$ of such that $\lim_\gamma \inf h_{\alpha_{\beta_\gamma}}(x) > -\frac{1}{l_0}$ on the set $A' \subset A_0$ such that $A' \in S \setminus I$.

Let $x \in A'$, Then

$$\begin{aligned}
\limsup_\alpha g_\alpha(x) &\ge \limsup_\gamma g_{\alpha_{\beta_\gamma}}(x) = \limsup_\gamma (f_{\alpha_{\beta_\gamma}}(x) + h_{\alpha_{\beta_\gamma}}(x)) \\
&\ge \limsup_\gamma (f_{\alpha_{\beta_\gamma}}(x) + \liminf_\gamma h_{\alpha_{\beta_\gamma}}(x)) \\
&= \limsup_\gamma f_{\alpha_{\beta_\gamma}}(x) + \liminf_\gamma h_{\alpha_{\beta_\gamma}}(x) \\
&> \frac{1}{k_0} - \frac{1}{l_0} > 0.
\end{aligned}$$

From the arbitrarines of $\{g_\alpha\}_{\alpha<\mathfrak{m}}$ and from the definition of U we have $U(x) > \frac{1}{k_0}$. I-a.e. on A'. This contradiction ends the proof.

Let (X, τ) be a topological space with N a family of nowhere dense sets. Let $\beta_0 = \min\{\beta : \bigcup_{\alpha<\beta} P_\alpha = \Theta \in \tau, P_\alpha \in N\}$. We choose a $\aleph_0 \le \beta \le \beta_0$ and say that a set A is of the β-first category if it is the union of family of cardinality β of sets from N. The family of sets of the β-first category is a β-ideal. We denote it by I_β. We shall say that a set A has the property of β-Baire (is a β -Baire or B_β set) if it can be represented in the form $A = G \triangle P$, where $G \in \tau$ and $P \in I_\beta$.

Lemma 4. *A set A has the property of β-Baire if and only if it can be represented in the form $A = \mathcal{F}\triangle Q$, where $\mathcal{F}$ is closed and Q is of β-first category.*

If A has the property of β-Baire, then so does it complement.

The class of sets having the property of β-Baire is a β-algebra. It is β-algebra generated by the open sets together with the sets of β-first category.

A set has the property of β-Baire if and only if it can be represented as a G_δ^β set plus a set of β-first category (or as an $\mathcal{F}_\sigma^\beta$ set minus a set of β-first category).

Any set having the property of β-Baire can be represented in the form $A = G\triangle P$, where G is a regular open set and P is of β-first category. This representation is unique.

Proof. One can follow exactly the proofs of Theorems 4.1-4.4 and 4.6 in [4].

From Lemma 4 as $\beta < \beta_0$ we have

Lemma 5. *The quotient Boolean algebra B_β/I_β is isomorphic to the algebra $RO(X,T)$ of regular open subsets of X.*

Proof. One can follow exactly proof of Theorem 4 [2].

Corollary 2. *B_β/I_β is a complete algebra.*

Remark. If we assume additionaly that Suslin's number of (X,τ) is β then algebra B_β/I_β fulfils β-chain condition.

Remark. For any $\aleph \leq \alpha, \beta < \beta_0$ we have algebras B_α/I_α and B_β/I_β to be isomorphic.

Example. Let $\mathfrak{k}$ - be infinite cardinal number. Let us say after [3] (p. 47) for x, $y \subset \mathfrak{k}$, $\operatorname{card} x = \mathfrak{k}$, $\operatorname{card} y = \mathfrak{k}$, that x and y are almost disjoint if $\operatorname{card}(x \cap y) < \mathfrak{k}$. A family $A \subset 2^{\mathfrak{k}}$ is called almost disjoint if any two elements of A are almost disjoint. An almost disjoint family $A \subset 2^{\mathfrak{k}}$ is called maximal if no element of $2^{\mathfrak{k}} \setminus A$ is almost disjoint with every member of A. In the sequel we shall need the following theorem from [3] (p. 48)

Let $\mathfrak{k} \geq \omega$ be a regular cardinal number.

a) *if $A \subset 2^{\mathfrak{k}}$ is almost disjoint family and* $\mathrm{card}(A) = \mathfrak{k}$ *then A is not maximal.*

b) *there is a maximal almost disjoint family $B \subset 2^{\mathfrak{k}}$ of power $\geq \mathfrak{k}^+$.*

Let $X = \{\alpha : \alpha < \omega_1\}$. We have $\mathrm{card}\, X = \aleph_1$.

The maximal almost disjoint family $P \subset 2^X$ is now of cardinality $> \aleph_1$. Let Y be arbitrary set $\mathrm{card}(Y) = \mathrm{card}(P)$.

Then there is an one-to-one and onto function $\varphi : P \longrightarrow Y$. The family $S = 2^Y$ is an $\aleph_1$-algebra of sets and $I = \{A \subset Y : \mathrm{card}(A) \leq \aleph_1\}$ is a proper $\aleph_1$-ideal. We define now the $\aleph_1$-sequence $\{f_\alpha\}_{\alpha<\omega_1}$, $f_\alpha : Y \longrightarrow \{O, 1\}$ as follows

$$f_\alpha(y) = \begin{cases} 0 & \text{for } \alpha \notin \varphi^{-1}(y) \\ 1 & \text{for } \alpha \in \varphi^{-1}(y) \end{cases}$$

We have both $\varphi^{-1}(y)$ and $X - \varphi^{-1}(y)$ to be cofinal with X for every $y \in Y$ and therefore $\{f_\alpha\}$ is not convergent at any y so it is not I-a.e. convergent on Y.

On the other side let $\{f_{\alpha_\beta}\}_{\beta<\omega_1}$ be arbitrary subsequence of $\{f_\alpha\}_{\alpha<\omega_1}$. The family P is maximal so there is $p_0 \in P$ such that $p_0 \cap \{\alpha_\beta\}_{\beta<\omega_1}$ is cofinal with ω_1. For $\{\alpha_\beta\}_{\beta<\omega_1}$ we define subsequence $\{\alpha_{\beta_\gamma}\}_{\gamma<\omega_1} = p_0 \cap \{\alpha_{\beta<\omega_1}\}$. We have $\lim_{\gamma\to\omega_1} f_{\alpha_{\beta_\gamma}}(y) = 1$ for $y_0 = \varphi(p_0)$ and $\lim_{\gamma\to\omega_1} f_{\alpha_{\beta_\gamma}}(y) = 0$ for $y \neq y_0$ as p_0 and $p = \varphi^{-1}(y)$ are almost disjoint. The sequence $\{f_\alpha\}_{\alpha<\omega_1}$ is therefore convergent with respect to ω_1-ideal I to function equal zero on Y.

Let us consider three families of sets $\tau_i = \{A \subset Y : \mathrm{card}(Y - A) < \omega_i) < \omega\} \cup \emptyset$, $i = 0, 1, 2$. Each family is the topology on Y. The ideal of nowhere dense sets is here of the form $A = \{A \subset Y : \mathrm{card}\, A < \omega_i\}$. For any i the family $\{A \subset Y : A = \bigcup_{\gamma<\omega} A_\gamma,\ A_\gamma \in A_i\}$ is precisely the ideal I_{ω_1} of ω_1-first category sets. The sequence $\{f_\alpha\}_{\alpha<\omega_1}$ given above is now convergent in general with respect to I_{ω_1} but not I_{ω_1}-a.e. convergent (to function equal to zero on Y).

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GRANICE GÓRNE I DOLNE WZGLĘDEM IDEAŁU $\mathfrak{m}$-ADDYTYWNEGO

W pracy rozważamy pozaskończone ciągi funkcji mierzalnych względem $\mathfrak{M}$-addytywnego ciała zbiorów. Określamy zbieżność takich ciągów oraz granice górną i dolną względem $\mathfrak{M}$-ideału zbiorów i pokazujemy, że ciąg jest zbieżny wtedy i tylko wtedy, gdy te granice sa równe. Badamy istotność założenia o mierzalności funkcji. Otrzymane wyniki są przeniesieniem twierdzeń E. Wagner ([7]) na przypadek ciągów pozaskończonych.

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