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HOLOMORFIC SOLUTION OF NONLINEAR GENERALIZED DIFFERENTIAL EQUATION

In this paper we study the problem of existence and uniqueness of holomorphic solution of the equation $Df(z)(h(z)) = F(z, f(z))$ for $z \in B_r^n$ with the condition $f(0) = 0$ and the assumption that 0 is a singular point (i.e. $h(0) = 0$).

Let $\mathbb{C}^n$ denote the space of n complex variables $z = (z_1, \ldots, z_n)$ with Euclidean inner product $< z, w >= \sum_{j=1}^{n} z_j w_j$ and norm $\|z\| = \sqrt{< z, z >}$. The ball $\{z; \|z\| < r\}$ is denoted by B_r^n. The class of holomorphic mappings from an open set Ω into $\mathbb{C}^n$ is denoted by $\mathcal{H}(\Omega, \mathbb{C}^n)$. The letter $\Im$ represent the identity map on $\mathbb{C}^n$. Let $h \in \mathcal{H}(B_r^n, \mathbb{C}^n), F \in \mathcal{H}(B_r^n \times B_\rho^n, \mathbb{C}^n), h(0) = 0$ and $F(0,0) = 0$. The considerations concerning existence and uniqueness mapping $f \in \mathcal{H}(B_{r_0}^n, \mathbb{C}^n)$ satisfying nonlinear generalized differential equation of the form

$$Df(z)(h(z)) = F(z, f(z)) \quad \text{for} \quad z \in B_{r_o}^n$$

(compare [6], [7]) are presented bellow. Let

$$\mathcal{M}_r = \{h \in \mathcal{H}(B_r^n, \mathbb{C}^n); h(0) = 0, Dh(0) = \Im, \mathrm{re} < h(z), z >> 0 \quad \text{for} \quad z \in B_r^n \setminus \{0\}\}.$$

Theorem 1. *Let* $h \in \mathcal{H}(B_r^n, \mathbb{C}^n), F \in \mathcal{H}(B_r^n \times B_\rho^n, \mathbb{C}^n), h(0) = 0, Dh(0) = \Im, F(0,y) = 0$ *for* $y \in B_\rho^n$. *Let* r_1, r_2, C, L *be positive constants such that*

(i) $$0 < r_1 < r, \quad 0 < r_2 < \rho,$$

(ii) $$\|F(z,y)\| \le C \quad \text{for} \quad (z,y) \in B_{r_1}^n \times B_{r_2}^n,$$

(iii) $$\|F(z,y_1) - F(z,y_2)\| \le L\|y_1 - y_2\| \quad \text{for} \quad z \in B_{r_1}^n, y_1, y_2 \in B_{r_2}^n,$$

(iv) $$h \in \mathcal{M}_{r_1}.$$

Then for any r_0 *such that*

$$0 < r_0 < \min(r_1 + \frac{C}{2r_2} - \sqrt{\frac{r_1 C}{r_2} + \frac{C^2}{4r_2^2}}, r_1 + \frac{L}{2} - \sqrt{r_1 L + \frac{L^2}{4}}$$

the differential equation

(1) $$Df(z)(h(z)) = F(z, f(z)) \quad \text{for} \quad z \in B_{r_0}^n$$

with the condition $f(0) = 0$ *has exactly one solution* $f \in \mathcal{H}(B_{r_0}^n, B_{r_2}^n)$.

Proof. We first observe that by Theorem 2.1 from [5] and by $h \in \mathcal{M}_{r_1}$ the differential equation

(2) $$\frac{\partial v}{\partial t}(z,t) = -h(v(z,t))$$

has exactly one solution $v = v(z,t)$ defined for $(z,t) \in B_{r_1}^n \times [0,\infty)$. From theorems concerning dependence of solution of differential equation upon initial conditions (compare e.g. [1]) it follows that v is continuous on $B_{r_1}^n \times [0,\infty)$ and, for any $t \in [0,\infty), v(\cdot,t) \in \mathcal{H}(B_{r_1}^n, \mathbb{C}^n)$.

Next, let $\mathcal{H}_0^\infty$ denote the space of all holomorphic bounbed mappings f from B_r^n into $\mathbb{C}^n$, such that $f(0) = 0$, with the sup norm and a closed ball, in $\mathcal{H}_0^\infty$, with radius r and centre 0 will be denoted by K_r.

Consider the mapping T defined on K_{r_2} in the following way

$$T(f)(z) = \int_0^\infty F(v(z,t), f(v(z,t)))dt \quad \text{for} \quad z \in B_{r_0}^n$$

where $f \in K_{r_2}$ and v is the solution of (2).

We next prove that such definition T is correct and $T(K_{r_2}) \subset K_{r_2}$. Let us first observe that for any $y \in B^n_{r_2}$ the mapping $F(\cdot, y)$ satisfies the assumptions of the Schwarz Lemma (see [4], Theorem 7.19, p.56). Hence

$$\|F(z,y)\| \le C\frac{\|z\|}{r_1} \quad \text{for} \quad (z,y) \in B^n_{r_1} \times B^n_{r_2}. \tag{3}$$

By Lemma 2.2 from [5] we get immediately

$$\|v(z,t)\| \le \frac{r_1 r_0 e^{-t}}{(r_1 - r_0)^2} \quad \text{for} \quad (z,t) \in B^n_{r_0} \times [0,\infty). \tag{4}$$

Consequently, from (3) and (4) and by definition of r_0 we obtain

$$\|F(v(z,t), f(v(z,t)))\| \le \frac{Cr_0}{(r_1 - r_0)^2} e^{-t} \quad \text{for} \quad (z,t) \in B^n_{r_0} \times [0,\infty).$$

By the above, it follows that the definition of T is correct and $T(K_{r_2}) \subset K_{r_2}$. We now show that the mapping T is contractive. Using the Schwarz Lemma and our assumptions about F we have

$$\|T(f_1)(z) - T(f_2)(z)\| \le \frac{L}{r_1}\|z\|\|y_1 - y_2\| \quad \text{for} \quad z \in B^n_{r_1}, y_1, y_2 \in B^n_{r_2}. \tag{5}$$

Let $f_1, f_2 \in K_{r_2}$, then from (4) and (5) it follows that

$$\|T(f_1)(z) - T(f_2)(z)\| \le L\int_0^\infty \frac{r_0 e^{-t}}{(r_1 - r_0)^2}\|f_1(v(z,t)) - f_2(v(z,t))\|dt$$

for $z \in B^n_{r_1}$. Hence

$$\|T(f_1) - T(f_2)\| \le \frac{Lr_0}{(r_1 - r_0)^2}\|f_1 - f_2\| \quad \text{for} \quad f_1, f_2 \in K_{r_2}$$

and in consequence T is contractive.

The Banach contraction principle (see e.g. [2], Theorem 1.1) yields that there exists exactly one the mapping $f_0 \in K_{r_2}$ which is a fixed point of T. Now we show that f_0 is a solution of (1). By the definition of f_0 we have

$$f_0(z) = \int_0^\infty F(v(z,t), f_0(v(z,t)))dt \quad \text{for} \quad z \in B^n_{r_0}.$$

Since $v(v(z,t),s) = v(z,t+s)$ for $s,t \in [0,\infty)$ and $z \in B^n_{r_0}$ we conclude that

$$(6) \qquad f_o(v(z,s)) = \int_s^\infty F(v(z,t), f_o(v(z,t)))dt$$

for $s \in [0,\infty)$ and $z \in B^n_{r_0}$. Differentiating both sides of equality (6) with respect to the parameter s we obtain for $s = 0 Df_0(z)(-h(z)) = -F(z, f_0(z))$ for $z \in B^n_{r_0}$.

Hence, the mapping $f_0 : B^n_{r_0} \longrightarrow B^n_{r_2}$ is a holomorphic solution of equation (1) satisfying condition $f_0(0) = 0$. The next theorem also gives a sufficient condition for the existence and uniqueness of solution of equation (1).

Theorem 2. *Let $h \in \mathcal{H}(B^n_r, \mathbb{C}^n), F \in \mathcal{H}(B^n_r \times B^n_\rho, \mathbb{C}^n)$ be such that $h(0) = 0, Dh(0) = \Im, F(0,y) = y$ and $D_1F(0,y) = 0$ for $y \in B^n_\rho$. Let r_1, r_2, C, L be positive constans such that*

(a) $\quad 0 < r_1 < r, \quad 0 < r_2 < \rho,$

(b) $\quad \|F(z,y) - y\| \le C \quad \text{for} \quad (z,y) \in B^n_{r_1} \times B^n_{r_2},$

(c) $\quad \|F(z,y_1) - F(z,y_2) - y_1 + y_2\| \le L\|y_1 - y_2\| \quad \text{for} \quad z \in B^n_{r_1}, y_1, y_2 \in B^n_{r_2}.$

Then for any r_0 such that $0 < r_0 < \min(r_1, \alpha, \beta, \gamma)$ where

$$\alpha = \frac{1}{2}\left(2r_1 + \sqrt{L} - \sqrt{4r_1\sqrt{L} + L}\right),$$

$$\beta = \frac{1}{2}\left(2r_1 + \frac{r_1}{r_2} - \sqrt{\frac{4r_1^2}{r_2} + (\frac{r_1}{r_2})^2}\right)$$

$$\gamma = \frac{1}{2}\left(2r_1 + \frac{1}{t_0} - \sqrt{(2r_1 + \frac{1}{t_0})^2 - 4r_1^2}\right),$$

$$t_0 = \frac{-r_1 + \sqrt{r_1^2 + 4Cr_2}}{2C},$$

the differential equation

$$Df(z)(h(z)) = F(z, f(z)) \quad \textit{for} \quad z \in B^n_{r_0}$$

with the coditions $f(0) = 0, Df(0) = \Im$ *has exactly one solution* $f \in \mathcal{H}(B^n_{r_0}, B^n_{r_2})$.

Proof. Let, as in the proof of theorem 1, $v = v(z,t)$, for $(z,t) \in B^n_{r_1} \times [0, \infty)$, be a solution of equation (2). By Theorem 2 from [3] the function g defined by equality

$$g(z) = \lim_{t \to \infty} (e^t v(z,t)) \quad \text{for} \quad z \in B^n_r$$

belongs to $\mathcal{H}(B^n_{r_1}, \mathbb{C}^n)$. Let $\mathcal{H}_0^\infty$ be defined as in the proof of the previous theorem and let $K(g, \tau)$ denote a closed ball, in $\mathcal{H}_0^\infty$, with radius τ and centre g. Assume that $0 < \tau < r_2 - \frac{r_1 r_0}{(r_1 - r_0)^2}$. Next, consider the integral operator T of the form

$$T(f)(z) = g(z) + \int_0^\infty e^t [F(v(z,t), f(v(z,t))) - f(v(z,t))] dt$$

for $z \in B^n_{r_0}$ and $f \in K(g, \tau)$. Observe that by Theorem 7.19 from [4]

$$\|F(z,y) - y\| \le C \frac{\|z\|^2}{r_1^2} \quad \text{for} \quad (z,y) \in B^n_{r_1} \times B^n_{r_2}. \tag{7}$$

Consequently, from (7) and (4) we have

$$\|F(v(z,t), f(v(z,t))) - f(v(z,t))\| \le C \frac{r_0^2 e^{-2t}}{(r_1 - r_0)^4} \tag{8}$$

for $(z,t) \in B^n_{r_0} \times [0,\infty)$. By the definition of g and by (4) we get

$$\|g(z)\| \le \frac{r_1 r_0}{(r_1 - r_0)^2} \quad \text{for} \quad z \in B^n_{r_0}.$$

From the above inequality, the definition r_0 and by inequality (8) it follows that the mapping T is correctly defined and maps $K(g,\tau)$ into $K(g,\tau)$. Now we show that T is contractive. Using the Schwarz Lemma and our assumptions about F we obtain

$$\|F(z,y_1) - F(z,y_2) - y_1 + y_2\| \le L\|y_1 - y_2\|\frac{\|z\|^2}{r_1^2}$$

for $z \in B^n_{r_1}$ and $y_1, y_2 \in B^n_{r_2}$. From this and (4) we have

$$\text{(9)} \qquad \|T(f_1)(z) - T(f_2)(z)\| \le L\int_0^\infty \frac{r_0^2 e^{-t}}{(r_1 - r_0)^4}\|f_1(v(z,t)) - f_2(v(z,t))\|dt$$

for $z \in B^n_{r_0}$ and $f_1, f_2 \in K(g,\tau)$. Since $\frac{Lr_0^2}{(r_1-r_0)^4} < 1$, therefore from (9) it follows immediately that the mapping T is contractive. Hence, by the Banach contraction principle, there exists exactly one $f_0 \in K(g,\tau)$ being a fixed point of the mapping T. Next, we prove that $f_0$is a solution of (1).By the definition of f_0 we have

$$\text{(10)} \qquad f_0(z) = g(z) + \int_0^\infty e^t[F(v(z,t), f_0(v(z,t))) - f_0(v(z,t))]dt$$

for $z \in B^n_{r_0}$. Since $v(v(z,t),s) = v(z,t+s)$ for $s,t \in [0,\infty)$ and $z \in B^n_{r_0}$ therefore from (10) it follows that

$$f_0(v(z,s)) = g(v(z,s)) + \int_s^\infty e^{t-s}[F(v(z,t), f_0(v(z,t))) - f_0(v(z,t))]dt$$

for $s \in [0,\infty)$ and $z \in B^n_{r_0}$. Differentiating both sides of this equality with respect to the parameter s we obtain for $s = 0$

$$\begin{aligned} Df_0(z)(-h(z)) &= Dg(z)(-h(z)) \\ &- \int_0^\infty e^t[F(v(z,t), f_0(v(z,t))) - f_0(v(z,t))]dt \\ &- F(z, f_0(z)) + f_0(z) \end{aligned}$$

for $z \in B^n_{r_0}$. As $Dg(z)(h(z)) = g(z)$ for $z \in B^n_{r_0}$ (compare [3], Theorem 4), the above equality and (10) gives that f_0 is a solution of (1).

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O HOLOMORFICZNYCH ROZWIĄZANIACH UOGÓLNIONYCH RÓWNAŃ RÓŻNICZKOWYCH

W tej pracy badane jest istnienie i jednoznaczność holomorficznego rozwiązania równania $Df(z)(h(z)) = F(z, f(z))$ dla $z \in B_n^r$ przy warunku $f(0) = 0$ i przy za/łożeniu, że 0 jest punktem osobliwym (tzn. $h(0) = 0$).

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