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ON DENSITY TOPOLOGIES GENERATED BY IDEALS

We discuss some properties of the density topology, generated by a given ideal I, in connection with the countable chain condition. Namely, we prove that for every finite family of invariant σ–algebras with invariant σ–ideals, satisfying the countable chain condition, there exists an element of the density topology, which is not measurable with respect to all of these σ–algebras. In particular, we obtain a generalization of one result given in [5].

Let $\mathbb{R}$ be the real line equipped with the standard Euclidean topology. Denote by l the usual Lebesgue measure on $\mathbb{R}$. Let X be an arbitrary Lebesgue measurable subset of $\mathbb{R}$ and let x be an arbitrary point of $\mathbb{R}$. Take any $h > 0$ and consider the real number

$$d(X, x, h) = l(X \cap [x - h, x + h])/2h.$$

Suppose that

$$lim_{h \to 0}\ d(X, x, h)$$

there exists and denote this limit by $d(X, x)$. The real number $d(X, x)$ is called a density of the set X at the point x. If the equality $d(X, x) = 1$ holds, then the point x is called a Lebesgue density point of X. The classical Lebesgue theorem states that almost all points of the set X are its density points. We may put

$$\Phi_d(X) = \{x \in \mathbb{R} : d(X, x) = 1\}$$

and we may consider the following class of sets:

$$T_d = \{X \in dom(l) : X \subseteq \Phi_d(X)\}.$$

It is well-known that the class T_d is a topology on the set $\mathbb{R}$, extending the Euclidean topology of $\mathbb{R}$. The topology T_d is called the density topology of $\mathbb{R}$. It is worth remarking here that the density topology was intensively investigated by many authors (see, for example, [1] where some interesting properties of this topology are considered; see also [2] where some additional properties of the density topology are discussed as well).

It is easy to check that the zero–point 0 is a density point of a Lebesgue measurable set $X \subseteq \mathbb{R}$ if and only if the relation

$$lim_{n\to\infty}\, n \cdot l(X \cap [-1/n,\, 1/n]) = 2$$

holds. Obviously, this relation is equivalent to the equality

$$lim_{n\to\infty}\, l(nX \cap [-1,\, 1]) = 2,$$

where nX denotes the set $\{nx : x \in X\}$. The last equality means that the sequence of characteristic functions

$$\{\chi_{nX\cap[-1,1]} : n \in \mathbb{N}\}$$

converges in measure to the characteristic function of the unit segment $[-1,\, 1]$. Now, applying the well–known Riesz theorem from the classical measure theory, we can describe the convergence in measure in terms of the convergence almost everywhere. This simple (but important) observation is due to W.Wilczynski who introduced in 1985 the concept of a density point with respect to category (see [3] and [4]).

Let T be a topology on $\mathbb{R}$ satisfying the following three conditions:

(1) the unit segment $[-1,\ 1]$ has the Baire property with respect to T;

(2) for every natural number n and for every set X having the Baire property with respect to T, the set nX has also the Baire property with respect to T;

(3) for every translation g of $\mathbb{R}$ and for every set X having the Baire property with respect to T, the set $g(X)$ has also the Baire property with respect to T.

Note that these conditions are sufficient to introduce the concept of a density point in the category sense.

Let $B(\mathbb{R},T)$ denote the σ–algebra of all sets having the Baire property with respect to the topology T and let $K(\mathbb{R},T)$ denote the σ–ideal of all first category sets with respect to T. Furthermore, let X be an arbitrary set from the σ–algebra $B(\mathbb{R},T)$. We say that the zero–point 0 is a $K(\mathbb{R},T)$–density point of the set X if the sequence of characteristic functions

$$\{f_n : n \in \mathbb{N}\} = \{\chi_{nX\cap[-1,1]} : n \in \mathbb{N}\}$$

is convergent to the characteristic function $\chi_{[-1,1]}$ with respect to the σ–ideal $K(\mathbb{R},T)$. The last sentence means that 0 is a $K(\mathbb{R},T)$–density point of the set X if and only if for each infinite subset N_1 of $\mathbb{N}$ there exists an infinite subset N_2 of N_1 such that the corresponding partial sequence of functions $\{f_n : n \in N_2\}$ is convergent pointwise to $\chi_{[-1,1]}$ on the complement of a member of $K(\mathbb{R},T)$.

Now, let x be an arbitrary point of $\mathbb{R}$ and let X be an arbitrary set from the σ–algebra $B(\mathbb{R},T)$. We say that x is a $K(\mathbb{R},T)$–density point of the set X if 0 is a $K(\mathbb{R},T)$–density point of the translated set

$$X - x = \{y - x : y \in X\}.$$

Futhermore, for any set X having the Baire property with respect to the topology T, let us put

$\Phi_T(X) =$ the set of all $K(\mathbb{R},T)$–density points of X.

Hence, we obtain the following family of sets:

$$T^* = \{X \in B(\mathbb{R},T)\ :\ X \subseteq \Phi_T(X)\}.$$

If the family T^* forms a topology on the basic set $\mathbb{R}$, then T^* is called the Wilczyński topology on $\mathbb{R}$, associated with the original topology T (note that T^* is also called the $K(\mathbb{R}, T)$-density topology on $\mathbb{R}$).

Thus, we have the concept of a density topology on $\mathbb{R}$ in the sense of category. Of course, an analogous definition can be formulated for an arbitrary group G equipped with a topology T satisfying the conditions similar to conditions (1) – (3).

Example 1. Let us consider a particular case of the above construction. Namely, let us take as T the standard Euclidean topology on $\mathbb{R}$. In this case it can be shown that T^* is a topology on $\mathbb{R}$ called the I-density topology (see [3], [4], and [5]). The I-density topology has a number of interesting properties and it was investigated by many authors. Among various works devoted to this topology we mention especially the book [5] where the class of all continuous functions with respect to the I-density topology is studied in details.

Example 2. Let us consider another particular case of the Wilczyński construction. Namely, let us put $T = T_d$, where T_d is the density topology on $\mathbb{R}$. Obviously, T satisfies conditions (1) – (3). It is easy to check that the equality

$$\Phi_T(X) = \Phi_d(X)$$

holds for each subset X of the real line having the Baire property with respect to T. Thus, we see that

$$T^* = T = T_d,$$

i.e. the density topology T_d can be considered as a particular case of the Wilczyński topology.

The next example is a generalization of Example 2.

Example 3. Let μ be an arbitrary measure on the real line, satisfying the following relations:

1) μ is a complete measure;
2) μ is an extension of the Lebesgue measure l on $\mathbb{R}$.

Furthermore, let X be a μ–measurable subset of $\mathbb{R}$ and let x be a point of $\mathbb{R}$. We say that x is a μ–density point of the set X if the equality

$$lim_{n\to\infty}\, n \cdot \mu(X \cap [x-1/n, x+1/n]) = 2$$

holds. It is clear that if $\mu = l$, then this definition gives us the classical definition of a Lebesgue density point.

For each set $X \in dom(\mu)$, let us put

$\Phi_\mu(X) =$ the set of all μ–density points of X.

Now we may consider the family of sets

$$T_\mu = \{X \in dom(\mu) \,:\, X \subseteq \Phi_\mu(X)\}.$$

It can easily be shown that the family T_μ is a topology on the basic set $\mathbb{R}$. Moreover, in [6] is established, using the classical Vitali covering theorem, that for any set $X \in T_\mu$ there always exist subsets L, Y and Z of $\mathbb{R}$ such that

$$L \in dom(l),\ \mu(Y) = \mu(Z) = 0,\ X = (L \cup Y) \setminus Z.$$

We see also that the topology T_μ extends the usual density topology T_d and, if $\mu = l$, then T_μ coincides with T_d.

Suppose now that our measure μ satisfies relations 1), 2) and the following two relations:

3) μ is invariant under the group of all translations of $\mathbb{R}$;

4) for every natural number n and for every μ–measurable set X, the set nX is μ–measurable, too, and

$$\mu(nX) = n \cdot \mu(X).$$

Then it is easy to see that the notion of a μ–density point can be formulated in terms of the σ–ideal of all μ–measure zero sets (by the general scheme of Wilczyński considered above). Thus, we conclude that if relations 1)–4) hold for the given measure μ, then the topology T_μ can be obtained by the scheme of Wilczyński.

Note that the Wilczyński construction can be applied in a more general case (see, e.g., [5]). Namely, let I be a fixed ideal of subsets of the real line $\mathbb{R}$. Let $\{f_n : n \in \mathbb{N}\}$ be a sequence of functions acting from $\mathbb{R}$ into $\mathbb{R}$. We say that this sequence converges (I) to a function $f : \mathbb{R} \to \mathbb{R}$ if for every infinite subset N_1 of $\mathbb{N}$ there exists an infinite subset N_2 of N_1 such that the partial sequence of functions $\{f_n : n \in N_2\}$ converges pointwise to f on the complement of a member from the ideal I.

We say that a point $x \in \mathbb{R}$ is an I–density point of a given set $X \subseteq \mathbb{R}$ if the sequence of characteristic functions

$$\{\chi_{n(X-x)\cap[-1,1]} : n \in \mathbb{N}\}$$

converges (I) to the characteristic function $\chi_{[-1,1]}$.

Denote by the symbol $\Phi_I(X)$ the set of all I–density points of the set X.

Now let us put

$$T_I = \{X \subseteq \mathbb{R} : X \subseteq \Phi_I(X)\}.$$

It is not difficult to check that the family T_I is a topology on the set $\mathbb{R}$. We say that T_I is the topology on $\mathbb{R}$ generated by the given ideal I.

Some general properties of the topology T_I are discussed in [5]. In connection with these properties a certain set $A \subset \mathbb{R}$ is constructed in [5], satisfying the following relations:

1) A is a Lebesgue non–measurable subset of $\mathbb{R}$;
2) A does not have the Baire property with respect to the Euclidean topology of $\mathbb{R}$;
3) for each point $a \in A$ the equality

$$lim_{n\to\infty}\chi_{n(A-a)\cap[-1,1]} = \chi_{[-1,1]}$$

holds; in particular, $A \in T_I$ for every ideal I of subsets of $\mathbb{R}$.

The construction of the set A mentioned above explores essentially in [5] the existence of a Hamel basis of $\mathbb{R}$ being also a Bernstein subset of $\mathbb{R}$.

In this paper we shall show that a much stronger result can be obtained. For this purpose we need some auxiliary notions.

Let S be a σ–algebra of subsets of the real line $\mathbb{R}$, let J be a σ–ideal of subsets of $\mathbb{R}$ and let $J \subset S$. Recall that the pair (J, S)

satisfies the countable chain condition if, for any uncountable family $\{X_\xi : \xi < \omega_1\}$ of pairwise disjoint sets from S, there exists a set X_ξ belonging to the σ–ideal J (from this definition it follows also that all sets X_ξ, except a countable number of them, belong to the σ–ideal J).

We say that a σ–algebra S (respectively, a σ–ideal J) is invariant under the group of all translations of $\mathbb{R}$ if for every set X from S (respectively, from J) and for every translation g of $\mathbb{R}$, the set $g(X)$ belongs to S (respectively, to J).

Now, let us consider the real line $\mathbb{R}$ as a vector space E over the field $\mathbb{Q}$ of all rational numbers. According to a well–known theorem of the theory of vector spaces, there exists a basis B of E (this basis is called a Hamel basis of E). For any element $e \in E$ we have the unique representation

$$e = q_1 b_1 + q_2 b_2 + ... + q_m b_m,$$

where $m = m(e)$ is a natural number, $q_1, q_2, ..., q_m$ are rational numbers and $b_1, b_2, ..., b_m$ are pairwise distinct elements of B.

Let us put

$$\|e\| = |q_1| + |q_2| + ... + |q_m|.$$

Obviously, the functional $\| \ \|$ is a norm on E with the values contained in $\mathbb{Q}$. Moreover, it is easy to see that $(E, \| \ \|)$ is a nonseparable normed vector space.

Let us take an arbitrary sequence

$$r_1, r_2, ..., r_k, ... (k \in \mathbb{N}, \ k > 0)$$

of strictly positive irrational numbers such that

$$lim_{k\to\infty} r_k = \infty.$$

Consider the family of sets $\{A_k : k \in \mathbb{N}, \ k > 0\}$, where

$$A_k = \{e \in E : \|e\| < r_k\}.$$

Obviously, each set A_k is an open ball in the space E and, since r_k is an irrational number, the set $E \setminus A_k$ is open in E, too. These properties of the set A_k immediately give us the following

Lemma 1. *For the set A_k and for each point a of A_k we have the equality*

$$lim_{n\to\infty}\ \chi_{n(A_k-a)\cap[-1,1]} = \chi_{[-1,1]}.$$

An analogous equality holds for the set $E \setminus A_k$ and for each point a of $E \setminus A_k$.

In particular, for every ideal I of subsets of $E = \mathbb{R}$ we have

$$A_k \in T_I,\ E \setminus A_k \in T_I.$$

A more detailed proof of this Lemma see in [5].

We need also the following

Lemma 2. *For each set A_k there exists an uncountable family $\{e_\xi\ :\ \xi < \omega_1\}$ of elements of E (certainly, depending on A_k) such that the family*

$$\{A_k + e_\xi\ :\ \xi < \omega_1\}$$

consists of pairwise disjoint sets.

The proof of this Lemma see in [7] where a more general result is established. Namely, in [7] is proved that if V is an arbitrary nonseparable normed vector space and Z is a countable union of balls in V whose radii are equal to a fixed number $r > 0$, then there exists an uncountable family $\{v_\xi\ :\ \xi < \omega_1\}$ of elements of V such that the family

$$\{Z + v_\xi\ :\ \xi < \omega_1\}$$

consists of pairwise disjoint sets. From this fact it follows also that the set Z is absolutely negligible in the space V (about the last notion see [6] or [7]).

Lemma 3. *Let $\{J_1,\ J_2, ...,\ J_p\}$ be a finite family of σ-ideals of subsets of $\mathbb{R}$, let $\{S_1,\ S_2, ...,\ S_p\}$ be a finite family of σ-algebras of subsets of $\mathbb{R}$, and suppose that the following relations hold:*

1) $J_1 \subset S_1$, $J_2 \subset S_2, ..., J_p \subset S_p$;

2) all pairs (J_1, S_1), $(J_2, S_2), ...,\ (J_p, S_p)$ satisfy the countable chain condition;

3) all classes of sets

$$J_1,\ J_2, ...,\ J_p,\ S_1,\ S_2, ...,\ S_p$$

are invariant under translations of $\mathbb{R}$.

Then there exists a set A_k *such that*

$$A_k \notin S_1 \cup S_2 \cup \ldots \cup S_p.$$

Proof. Suppose that, for any natural number $k > 0$, we have

$$A_k \in J_1 \cup J_2 \cup \ldots \cup J_p.$$

Denote by $m(k)$ a natural number from $[1, p]$ such that $A_k \in J_{m(k)}$. In this way we obtain a sequence

$$m(1),\ m(2),\ \ldots\ ,\ m(k),\ \ldots$$

of natural numbers belonging to the segment $[1, p]$. Hence, there exists an infinite strictly increasing sequence

$$k_1,\ k_2,\ k_3,\ \ldots$$

of natural numbers such that

$$m(k_1) = m(k_2) = m(k_3) = \ \ldots\ = \ m \in [1, p].$$

Therefore, the relations

$$A_{k_1} \in J_m,\ A_{k_2} \in J_m,\ A_{k_3} \in J_m, \ldots$$

are fulfilled. Since we have

$$lim_{k \to \infty}\, r_k \ = \ \infty,$$

and J_m is a σ–ideal of sets, we get

$$\mathbb{R} = A_{k_1} \cup A_{k_2} \cup A_{k_3} \cup \ldots \in J_m,$$

which is impossible. Hence, we can conclude that there exists at least one natural number $k > 0$ such that

$$A_k \notin J_1 \cup J_2 \cup \ldots \cup J_p.$$

Now, it is easy to show that for this number k we also have

$$A_k \notin S_1 \cup S_2 \cup ... \cup S_p.$$

Indeed, suppose that $A_k \in S_m$, where $m \in [1,p]$, and consider an uncountable family $\{e_\xi : \xi < \omega_1\}$ of elements of $E = \mathbb{R}$ described in Lemma 2. Since $A_k \in S_m \setminus J_m$ and the classes J_m and S_m are invariant under translations of $\mathbb{R}$, we deduce that all sets of the disjoint family

$$\{A_k + e_\xi : \xi < \omega_1\}$$

belong to $S_m \setminus J_m$, too. So, we see that the pair (J_m, S_m) does not satisfy the countable chain condition, which contradicts relation 2). Thus, the proof of Lemma 3 is complete.

Taking into account Lemmas 1, 2 and 3 we can formulate the following

Proposition. *Let I be an arbitrary ideal of subsets of the real line $\mathbb{R}$. Let $\{J_1, J_2, ..., J_p\}$ be a finite family of σ–ideals of subsets of $\mathbb{R}$ and let $\{S_1, S_2, ..., S_p\}$ be a finite family of σ–algebras of subsets of $\mathbb{R}$. Suppose also that relations* 1), 2) *and* 3) *of Lemma 3 are fulfilled for*

$$J_1, J_2, ..., J_p, S_1, S_2, ..., S_p.$$

Then there exists a subset A of $\mathbb{R}$ such that

(1) $A \in T_I$, $\mathbb{R} \setminus A \in T_I$;

(2) $A \notin S_1 \cup S_2 \cup ... \cup S_p$.

The proof of this Proposition can be deduced from the preceding lemmas without any difficulties. Indeed, we may put $A = A_k$ for a suitable natural number $k > 0$.

Example 4. Consider a particular case of the situation described above. Namely, let $p = 2$ and let

J_1 = the σ–ideal of all Lebesgue measure zero subsets of $\mathbb{R}$;

S_1 = the σ–algebra of all Lebesgue measurable subsets of $\mathbb{R}$;

J_2 = the σ–ideal of all first category subsets of $\mathbb{R}$;

S_2 = the σ-algebra of all subsets of $\mathbb{R}$ having the Baire property in $\mathbb{R}$.

Obviously, we have

$$J_1 \subset S_1, \ J_2 \subset S_2,$$

the pairs (J_1, S_1) and (J_2, S_2) satisfy the countable chain condition and the classes of sets

$$J_1, J_2, S_1, S_2$$

are invariant under the group of all translations of the real line. Furthermore, let I be an arbitrary ideal of subsets of the real line. Then, by our proposition, there exists a subset A of $\mathbb{R}$ such that

(1) $A \in T_I$, $\mathbb{R} \setminus A \in T_I$;

(2) $A \notin S_1 \cup S_2$.

Thus, we obtain the result from [5] mentioned above.

We see also that the topology T_I does not satisfy the countable chain condition.

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O ABSTRAKCYJNYCH TOPOLOGIACH GĘSTOŚCI

W pracy rozważa się pewne własności abstrakcyjnych topologii gęstości przy założeniu warunku przeliczalnego łańcucha. Udowodniono, że dla dowolnej skończonej rodziny niezmienniczych σ–ciał i σ–ideałów spełniających warunek przeliczalnego łańcucha istnieje element abstrakcyjnej topologii gęstości, który nie jest mierzalny względem każdego σ–ciała tej rodziny. W szczególności uzyskano uogólnienie rezultatu pracy [5].

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