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SOME REMARKS ON NONMEASURABLE
ALMOST INVARIANT SETS

We discuss some properties of almost invariant sets in connection
with a measure extension problem. We consider a question on the
existence of an almost invariant subset of a basic space, nonmea-
surable with respect to a given nonzero σ-finite quasiinvariant (in
particular, invariant) measure defined on this space.

Let E be a basic set and let G be a group of transformations of E.
In such situation we say that the pair (E, G) is a space equipped with
a transformation group. Suppose also that μ is a σ-finite measure
defined on a σ-algebra of subsets of E. We say that a subset X of
E is almost G-invariant with respect to the measure μ if, for every
transformation g from G, we have the equality

$$\mu(g(X)\triangle X) = 0$$

where the symbol $\triangle$ denotes the operation of the symmetric difference
of sets.

Notice that almost invariant sets play an important role in the gen-
eral ergodic theory and, in particular, in some questions concerning
extensions of quasiinvariant (respectively, invariant) measures (see,
for example, [1], [2] or [3]). For instance, the following auxiliary

proposition shows us that any σ-finite G-quasiinvariant (respectively, G-invariant) measure defined on the basic space E can be extended onto any almost G-invariant subset of E.

Lemma 1. Let μ be a σ-finite G-quasiinvariant (respectively, G-invariant) measure defined on a σ-algebra of subsets of E and let X be an almost G-invariant set with respect to μ. Then there exists a measure ν defined on some σ-algebra of subsets of E such that

1) ν is a G-quasiinvariant (respectively, G-invariant) measure;

2) ν extends μ;

3) X is a ν-measurable set, i.e. $X \in dom(\nu)$.

The proof of Lemma 1 is not difficult (see, for example, [2] or [3]). From this lemma we can conclude that, if a given almost invariant set X is nonmeasurable with respect to the original measure μ, then the measure ν strictly extends μ. So we see that the original nonzero σ-finite G-quasiinvariant (respectively, G-invariant) measure μ can be strictly extended provided that there exists at least one subset of the space E not belonging to $dom(\mu)$ and almost G-invariant with respect to μ. Therefore, the following question arises in a natural way.

Question 1. Let (E, G) be a basic space equipped with a transformation group. What are the necessary and sufficient conditions, formulated in terms of the pair (E, G), under which for every nonzero σ-finite G-quasiinvariant (respectively, G-invariant) measure μ defined on E there exists an almost G-invariant set with respect to μ not belonging to $dom(\mu)$?

Another, more interesting version of Question 1 is the following

Question 2. Let again (E, G) be a basic space equipped with a transformation group. What are the necessary and sufficient conditions, also formulated in terms of the pair (E, G), under which there exists a countable family $\{X_n : n \in \omega\}$ of subsets of E such that, for every nonzero σ-finite G-quasiinvariant (respectively, G-invariant) measure μ defined on E, at least one set X_n is almost G-invariant with respect to μ and does not belong to $dom(\mu)$?

These two questions are still open. In the present paper we concentrate our attention on Question 2. Namely, in the further considerations we will show that in some particular but important cases of (E, G) there exists a countable family of subsets of E which all are, in a certain sense, almost G-invariant with respect to each nonzero σ-finite G-quasiinvariant (G-invariant) measure μ defined on E and at least one of these subsets is nonmeasurable with respect to μ. We want to note here that the method used in the further considerations is taken from the work [4] (see also [2] and [3]).

We need one auxiliary notion from the topological measure theory. Let T be a topological space such that all one-element subsets of T are Borel sets in T. We say that T is a Luzin space if every σ-finite diffused Borel measure defined on T is identically equal to zero. Such spaces T are also called universally measure zero topological spaces. Notice that there are many interesting examples of uncountable universally measure zero subspaces of the real line $\mathbb{R}$. One of the earliest examples is due to Luzin. Namely, using the methods of the theory of analytic sets, Luzin constructed a subset Z of the real line satisfying the next two relations:

(1) $card(Z) = \omega_1$ where ω_1 denotes the first uncountable cardinal number;

(2) Z is a universally measure zero space with respect to the induced topology.

The construction of the mentioned set Z is given in detail in the well-known monograph of Kuratowski [5]. From the existence of the set Z we immediately obtain the following

Lemma 2. *Let Y be an arbitrary set of cardinality ω_1. Then there exists a σ-algebra S of subsets of Y satisfying the next conditions:*

1) *all one-element subsets of Y belong to S;*

2) *S is a countably generated σ-algebra;*

3) *every σ-finite diffused measure defined on S is identically equal to zero.*

We also need the following auxiliary proposition.

Lemma 3. Let E be a basic space with $card(E) = \omega_1$ and let G be a transitive group of transformations of E with $card(G) = \omega_1$. Then there exists a partition

$$\{E_\alpha : \alpha < \omega_1\}$$

of the space E such that

1) for each ordinal $\alpha < \omega_1$ the set E_α is at most countable;

2) for every subset A of ω_1 and for every transformation g from the group G the set

$$(g(\cup\{E_\alpha : \alpha \in A\}))\Delta(\cup\{E_\alpha : \alpha \in A\})$$

is at most countable, as well.

The proof of Lemma 3 is not difficult (see [2] or [3]). Notice that relation 2) of this Lemma shows us, in particular, that for every subset A of ω_1 the corresponding set $\cup\{E_\alpha : \alpha \in A\}$ is almost G-invariant with respect to any σ-finite diffused G-quasiinvariant (G-invariant) measure μ defined on the basic space E.

Taking into account the results of Lemmas 2 and 3, we obtain the following

Proposition 1. Let again E be a basic space of cardinality ω_1 and let G be a transitive group of transformations of E with the same cardinality. Then there exists a countable family $\{X_n : n \in \omega\}$ of subsets of E such that

1) for every σ-finite diffused G-quasiinvariant (G-invariant) measure μ defined on E each set X_n ($n \in \omega$) is almost G-invariant with respect to μ;

2) for every nonzero σ-finite G-quasiinvariant (G-invariant) measure μ defined on E at least one set X_n is nonmeasurable with respect to μ; moreover, there is an infinite number of sets from the family $\{X_n : n \in \omega\}$ which are nonmeasurable with respect to μ.

Proof. Indeed, applying the result of Lemma 2, let us take a countable family $\{A_n : n \in \omega\}$ of subsets of ω_1 satisfying the following conditions:

(1) σ-algebra S generated by this family contains all one-element subsets of ω_1;

(2) any σ-finite diffused measure defined on S is identically equal to zero.

Now, let us put

$$X_n = \cup\{E_\alpha : \alpha \in A_n\} \quad (n \in \omega).$$

Then it is easy to check that the family of sets $\{X_n : n \in \omega\}$ is a required one. We see also that for every nonzero σ-finite diffused measure μ defined on E at least one set X_n is nonmeasurable with respect to μ and, moreover, there is an infinite number of sets from the family $\{X_n : n \in \omega\}$ which are nonmeasurable with respect to μ.

Now, let us consider the case when $(E, G) = (\mathbb{R}, \mathbb{R})$. In other words, let us take the real line $\mathbb{R}$ equipped with the group of all its translations (of course, we can identify the additive group of $\mathbb{R}$ with the group of all translations of $\mathbb{R}$ by the canonical isomorphism between these two groups).

Using a Hamel base of $\mathbb{R}$, we can represent $\mathbb{R}$ as a direct sum

$$\mathbb{R} = G_1 + G_2 \quad (G_1 \cap G_2 = \{0\})$$

of two subgroups G_1 and G_2 in such a way that $card(G_1) = \omega_1$.

Denote by the symbol I the ideal of subsets of $\mathbb{R}$ generated by the family

$$\{Y + G_2 : Y \text{ is a countable subset of } G_1\}.$$

It is easy to check that

(1) I is a σ-ideal of subsets of $\mathbb{R}$;

(2) I is invariant under all translations of $\mathbb{R}$;

(3) for each set $Z \in I$ there exists an uncountable family $\{g_\alpha : \alpha < \omega_1\}$ of elements of the group G_1 such that

$$\{g_\alpha + Z : \alpha < \omega_1\}$$

is a pairwise disjoint family of sets.

From relation (3) we can conclude that for every σ-finite $\mathbb{R}$-quasi-invariant (respectively, $\mathbb{R}$-invariant) measure μ defined on $\mathbb{R}$ the equality

$$\mu_*(Z) = 0 \quad (Z \in I)$$

holds (the symbol μ_* denotes here the inner measure associated with μ).

Therefore, we have the following

Lemma 4. *Let μ be any σ-finite $\mathbb{R}$-quasiinvariant (respectively, $\mathbb{R}$-invariant) measure defined on $\mathbb{R}$. Then there exists a measure ν defined on $\mathbb{R}$ and satisfying the next relations:*

1) *ν is an $\mathbb{R}$-quasiinvariant (respectively, $\mathbb{R}$-invariant) measure;*

2) *ν extends μ;*

3) *$I \subseteq dom(\nu)$;*

4) *$\nu(Z) = 0$ for each set $Z \in I$.*

Now, let us consider the group G_1 as a basic space equipped with the group of transformations G_1. Since $card(G_1) = \omega_1$ and the group G_1 acts transitively on G_1, we can apply to G_1 the result of Proposition 1. According to this proposition there exists a countable family $\{Y_n : n \in \omega\}$ of subsets of G_1 such that

(a) for each index $n \in \omega$ and for each translation $g \in G_1$ we have

$$card((g + Y_n)\triangle Y_n) \leq \omega;$$

(b) for every nonzero σ-finite G_1-quasiinvariant (respectively, G_1-invariant) measure μ defined on G_1 at least one set Y_n is nonmeasurable with respect to μ; moreover, there is an infinite number of sets from the family $\{Y_n : n \in \omega\}$ which are nonmeasurable with respect to μ.

Let us put

$$X_n = Y_n + G_2 \quad (n \in \omega).$$

Then it is obvious that for each index $n \in \omega$ and for each translation $h \in \mathbb{R}$ we have

$$((h + X_n)\triangle X_n) \in I.$$

Of course, in this case we can not assert that the sets $X_n (n \in \omega)$ are almost $\mathbb{R}$-invariant in the basic space $\mathbb{R}$. Indeed, if l is the classical Lebesgue measure on $\mathbb{R}$ and the group G_2 is not a Lebesgue measure zero set in $\mathbb{R}$, then there exists a set X_n which is not almost $\mathbb{R}$-invariant with respect to l (moreover, there is an infinite number of sets from the family $\{X_n : n \in \omega\}$ which are not almost $\mathbb{R}$-invariant with respect to l). But, taking into account the properties of the ideal I mentioned above, we see that all sets X_n $(n \in \omega)$ are almost $\mathbb{R}$-invariant with respect to a certain measure l' which is defined on $\mathbb{R}$, extends l and is also an $\mathbb{R}$-invariant measure.

More generally, after the preceding remarks it is clear that we have the following result.

Proposition 2. *The countable family of sets* $\{X_n : n \in \omega\}$ *mentioned above satisfies the next relations:*

1) *for every σ-finite $\mathbb{R}$-quasiinvariant ($\mathbb{R}$-invariant) measure μ defined on $\mathbb{R}$ there exists an $\mathbb{R}$-quasiinvariant ($\mathbb{R}$-invariant) measure ν defined on $\mathbb{R}$ and extending μ such that all sets X_n $(n \in \omega)$ are almost $\mathbb{R}$-invariant with respect to ν;*

2) *for every nonzero σ-finite $\mathbb{R}$-quasiinvariant ($\mathbb{R}$-invariant) measure μ defined on $\mathbb{R}$ there exists at least one set X_n nonmeasurable with respect to μ; moreover, there is an infinite number of sets from the family $\{X_n : n \in \omega\}$ which are nonmeasurable with respect to μ.*

Remark 1. It is not difficult to see that a result analogous to Proposition 2 is valid for the m-dimensional Euclidean space $\mathbb{R}^m$ equipped with the group of all translations of $\mathbb{R}^m$ and, more generally, for an arbitrary uncountable vector space E equipped with the group of all translations of E. It can be also shown that the same result is true for some classes of uncountable groups equipped with the groups of all their left translations. But the following question is still open: let Γ be any uncountable group equipped with the group of all its left translations; is it true an analogue of Proposition 2 for Γ?

Remark 2. Let E be a basic space and let G be a group of transformations of E. Suppose that μ is a σ-finite G-quasiinvariant

(G-invariant) complete measure defined on a σ-algebra $S(\mu)$ of subsets of E. Let X be an arbitrary subset of E nonmeasurable and almost G-invariant with respect to μ. Then, according to Lemma 1, there exists a complete measure ν defined on a σ-algebra $S(\nu)$ of subsets of E and satisfying the following conditions:

1) ν is a G-quasiinvariant (G-invariant) measure;

2) ν strictly extends the original measure μ;

3) the set X belongs to $S(\nu)$.

Let us put

$I(\mu) = $ the ideal of all μ-measure zero sets;

$I(\nu) = $ the ideal of all ν-measure zero sets.

It is clear that we have two measure Boolean algebras

$$A(\mu) = S(\mu)/I(\mu), \quad A(\nu) = S(\nu)/I(\nu)$$

and, since ν extends μ, we have the canonical embedding

$$\phi : A(\mu) \to A(\nu).$$

It is not difficult to check that the measure ν can be taken in such a way that the mapping ϕ will not be a surjection. Hence, for this measure ν, the corresponding measure algebra $A(\nu)$ contains the measure algebra $A(\mu)$ as a proper subalgebra.

Finally, let us formulate the third proposition concerning the existence of nonmeasurable almost invariant subsets of a basic space E.

Proposition 3. *Let E be a nonempty basic set and let G be a group of transformations of E with $card(G) = \omega_1$, acting freely on E (in particular, from this condition it follows that E is an uncountable space). Then there exists a countable family $\{X_n : n \in \omega\}$ of subsets of E such that*

1) *for every nonzero σ-finite G-quasiinvariant (G-invariant) measure μ defined on E at least one set X_n is nonmeasurable*

with respect to μ; moreover, there is an infinite number of sets from this family which are nonmeasurable with respect to μ;

2) for every σ-finite G-quasiinvariant (G-invariant) measure μ defined on E there exists a G-quasiinvariant (G-invariant) measure ν defined on E extending μ and having the property that all sets X_n ($n \in \omega$) are almost G-invariant with respect to ν.

The proof of this proposition is similar (in some details) to the proof of Proposition 1.

Remark 3. Let (E, G) be a basic space equipped with a transformation group. It is not difficult to check that the following two sentences are equivalent:

1) all G-orbits in E are uncountable;

2) every σ-finite G-quasiinvariant (respectively, G-invariant) measure defined on E can be extended to a σ-finite diffused G-quasiinvariant (respectively, G-invariant) measure defined on E.

In particular, if E is an uncountable basic space and a group G of transformations of E acts transitively on E, then every σ-finite G-quasiinvariant (respectively, G-invariant) measure defined on E can be extended to a σ-finite diffused G-quasiinvariant (respectively, G-invariant) measure defined on E.

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PEWNE UWAGI O PRAWIE NIEZMIENNICZYCH ZBIORACH NIEMIERZALNYCH

W pracy rozważa się pewne własności prawie niezmienniczych zbiorów w kontekście problemu przedłużania miar niezmienniczych.

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