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ON THE TOPOLOGIES GENERATING
QUASI–ISOMORPHIC RINGS
OF CONTINUOUS FUNCTIONS

In this paper we show that two Tychonoff topologies on X are identical if and only if the rings of real continuouis functions corresponding to them are quasi-isomorphic.

In many papers the authors investigated the relations between the topological properties of spaces and the algebraic properties of rings of real contiuous functions defined on these spaces ([2], [6]). Many mathematicians studied the situations when, in a given space X for different topologies, the classes of real continuous functions were identical ([3], [4], [5]). In this context, it is purposeful to seek for as weak assumptions (of algebraic nature) as possible, concerning the rings of continuous functions, whose fulfilment would quarantee the equality of the topologies corresponding to them.

It is well know that, for different Tychonoff topologies, the classes of continuous functions, corresponding to them, are different; consequently, the equality of the classes of continuous functions for the Tychonoff topologies implies the identity of these topologies.

Therefore, it is natural in our investigations that we assume the topologies under considerations to be $T_{3\frac{1}{2}}$.

In this paper we use the standard notations (from monograph [1]).

If f is a real function defined on X, then we denote $Z(f) = \{x \in X : f(x) = 0\}$. Let (X, T_1), (X, T_2) denote topological spaces. We shall use the terminology, for example, T_1-neighbourhood or T_1-continuity, to make a distinction between the two topologies under consideration.

We say that the rings $C(T_1), C(T_2)$ of real continuous functions on (X, T_1), (X, T_2), respectively, are quasi- isomorphic if there exists a mapping $\Theta : C(T_1) \rightarrow C(T_2)$ such that Θ maps $C(T_1)$ onto $C(T_2)$ in a one-to-one way and $Z(f) = Z(\Theta(f))$ for each $f \in C(T_1)$.

Let T_1, T_2 be two Tychonoff topologies on a set X.

Theorem 1. *The topologies T_1 and T_2 are identical if and only if the rings $C(T_1)$ and $C(T_2)$ are quasi-isomorphic.*

Proof. Necessity. Of course, if $T_1 = T_2$, then $C(T_1) = C(T_2)$, and so, $C(T_1)$ and $C(T_2)$ are quasi- isomorphic.

Sufficiency. Let $\mathcal{A}_2$ denote the family of all sets A consisting of all functions $f \in C(T_2)$ such that $f(x_A) = 0$ for some fixed point x_A. This means that there exists a one-to-one correspondence between sets of the family $\mathcal{A}_2$ and points; precisely:

$$A \in \mathcal{A}_2 \text{ if and only if } f(x_A) = 0 \text{ for } f \in A.$$

Remark that: $x_A \neq x_B$ if and only if $A \neq B$; moreover, if $A, B \in \mathcal{A}_2$ and $A \neq B$, then A is not included in B and B is not included in A.

Define the neighbourhood system for $\mathcal{A}_2$ in the following way: for $C \in \mathcal{A}_2$, let $B(C)$ consists of all sets of the form $\{c\} \cup \{A \in \mathcal{A}_2 : t(x_A) \neq 0\}$ for some function $t \in C(T_2)$ such that $t(x_C) \neq 0$. It is easy to see that the collection $\{B(C)\}_{C \in \mathcal{A}_2}$ satisfies conditions (BP1)-(BP3) from monograph [1] (p.28).

Let now $\mathcal{A}_2$ denote the topological space with the topology generated by the neighbourhood system $\{B(C)\}_{C \in \mathcal{A}_2}$.

Let us define two mappings as follows:

$$\xi : (X, T_1) \rightarrow \mathcal{A}_2,$$

by the formula $\xi(x) = \{f \in C(T_2) : f(x) = 0\}$ and

$$\Phi : \mathcal{A}_2 \rightarrow (X, T_2),$$

by the formula $\Phi(A) = x_A$.

It is not difficult to see that ξ maps X onto $\mathcal{A}_2$ and Φ maps $\mathcal{A}_2$ onto X and both are one-to-one.

Now we shall show the continuity of Φ. Let $A \in \mathcal{A}_2$, $\Phi(A) = x_A$, and let W be an arbitrary T_2-neighbourhood of x_A. Since (X, T_2) is a Tychonoff space, then there exists $t \in C(T_2)$ such that $t(x_A) \neq 0$ and $t(x) = 0$ for each $x \notin W$.

Consider the neighbourhood U of A of the form

$$U = \{A\} \cup \{P \in \mathcal{A}_2 : t(x_P) \neq 0\}$$

Observe that $\Phi(U) \subset W$. Indeed, let $Q \in U$. In the case when $Q = A$, the above inclusion is obvious. Let then $Q \in U$ and $Q \neq A$. Thus $Q = \{f \in C(T_2) : f(x_Q) = 0\}$; moreover, $t(x_Q) \neq 0$. This means that $x_Q \in W$, and so, $\Phi(Q) = x_Q \in W$. This ends the proof of the continuity of Φ.

Now, we shall show that Φ^{-1} is continuous. Let $x \in X$ and let U be an arbitrary neighbourhood of $\Phi^{-1}(x)$. Then

$$U = \{f \in C(T_2) : f(x) = 0\} \cup \{A \in \mathcal{A}_2 : t(x_A) \neq 0\}$$

for some $t \in C(T_2)$ such that $t(x) \neq 0$.

From the T_2-continuity of t we infer that there exists a T_2- neighbourhood W of x such that $t(w) \neq 0$ for each $w \in W$. Then $\Phi^{-1}(W) \subset U$.

Now, we shall prove that ξ is continuous. Let $x \in X$ and let U be an arbitrary neighbourhood of $\xi(x)$. Then

$$U \supset \{f \in C(T_2) : f(x) = 0\} \cup \{A \in \mathcal{A}_2 : t(x_A) \neq 0\}$$

for some $t \in C(T_2)$ such that $t(x) \neq 0$. Let Θ be a quasi-isomorphism from $C(T_1)$ to $C(T_2)$ and let $t = \Theta(t_1)$ where $t_1 \in C(T_1)$. By the property of Θ, $t_1(x) \neq 0$. Let W be a T_1-neighbourhood of x such that $t_1(w) \neq 0$ for each $w \in W$. Observe that $\xi(W) \subset U$. Indeed, let $w \in W$ and $\xi(w) = \{f \in C(T_2) : f(w) = 0\} = P$. Since

$$P \longleftrightarrow x_p = w \quad \text{and} \quad t_1(w) \neq 0,$$

therefore $0 \neq \Theta(t_1)(w) = t(w)$. Thus $P \in \{A \in \mathcal{A}_2 : t(x_A) \neq 0\}$ and, consequently, $\xi(w) \in U$.

Now, we shall verify that the mapping ξ^{-1} is continuous. Let $A \in \mathcal{A}_2$ and $\xi^{-1}(A) = x_A$. Let W be an arbitrary T_1- neighbourhood of x_A. Since T_1 is a Tychonoff topology, there exists a function $t_1 \in C(T_1)$ such that

$$t_1(x_A) \neq 0 \ \text{ and } t_1(x) = 0 \ \text{ for each } x \notin W.$$

Put $t = \Theta(t_1)$. Thus

$$t(x_A) \neq 0 \ \text{ and } t(x) = 0 \ \text{ for each } x \notin W.$$

Consider a neighbourhood U of A defined as follows:

$$U = \{A\} \cup \{P \in \mathcal{A}_2 : t(x_P) \neq 0\}.$$

Then $\xi^{-1}(U) \subset W$.

To complete the proof, we remark that the composition $h = \Phi \circ \xi$ is a homeomorphism.

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O TOPOLOGIACH GENERUJĄCYCH QUASI-IZOMORFICZNE PIERŚCIENIE FUNKCJI CIĄGŁYCH

W pracy tej zostało pokazane, że dwie topologie Tichonowa określone na zbiorze X są identyczne wtedy i tylko wtedy, gdy odpowiadające im pierścienie funkcji ciągłych są quasi-izomorficzne.

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