ACTA UNIVERSITATIS LODZIENSIS
FOLIA MATHEMATICA 7, 1995

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# SMALL SETS IN UNCOUNTABLE ABELIAN GROUPS

**We discuss two kinds of small subsets of an uncountable Abelian group: negligible sets and absolutely negligible sets. We prove the existence of some partitions of a given uncountable Abelian group which consist of such small sets.**

In this paper we consider some finite and countable partitions of a given basic set $E$. These partitions consist of small subsets of $E$. Notice that here we discuss only two kinds of small subsets of $E$ which are closely connected with the general theory of transformation groups and quasi-invariant (in particular, invariant) measures with respect to such groups. Namely, we investigate here the so called negligible and absolutely negligible subsets of $E$. These notions were introduced and studied in the works [1] and [2].

Let $E$ be a non-empty basic space, let $\Gamma$ be a fixed group of transformations of $E$ and let $X$ be a subset of $E$. We say that the set $X$ is $\Gamma$-negligible in $E$ if the following two relations hold:

1) there exists a probability $\Gamma$-quasi-invariant measure $\mu$ defined on $E$ such that $X \in \mathrm{dom}(\mu)$;
2) for every probability $\Gamma$-quasi-invariant measure $\lambda$ defined on $E$, if $X \in \mathrm{dom}(\lambda)$ then $\lambda(X) = 0$.

There are many, essentially different from each other, examples of negligible subsets of $E$ (see [1] or [2]). For instance, if $E$ is the Euclidean plane and $\Gamma$ is the group of all translations of $E$ then the graph of any real function of a real variable is a $\Gamma$-negligible subset of $E$.

In connection with negligible sets a natural question arises: how small are these sets? A more concrete form of this question is the following: does there exist a finite partition of the space $E$ into $\Gamma$-negligible sets? It is clear that there does not exist a partition of $E$ into two $\Gamma$-negligible sets. On the other hand, we shall see below that some partitions of $E$ into three $\Gamma$-negligible sets are possible.

In further consideration we restrict ourselves to the case where the basic set $E$ is an uncountable Abelian group identified with the group $\Gamma$ of all its translations. This case is important for various applications. However, let us remark that in some situations considered below the condition of commutativity of the group $\Gamma$ is not necessary.

We need a number of auxiliary facts about the structure of infinite Abelian groups.

**Lemma 1.** *Let $(\Gamma, +)$ be an arbitrary uncountable Abelian group. Then there exist three subgroups $\Gamma_1$, $\Gamma_2$ and $\Gamma_3$ of $\Gamma$ such that*

1) *the cardinality of each of these subgroups is equal to $\omega_1$;*
2) *the sum $\Gamma_1 + \Gamma_2 + \Gamma_3$ is the direct sum of these subgroups.*

In connection with Lemma 1 notice that it can be proved starting from the well known theorems concerning the structure of infinite Abelian groups. Indeed, it is known that every uncountable Abelian group $\Gamma$ contains an uncountable subgroup $G$ which is the direct sum of a family of cyclic groups. But it is obvious that the group $G$ contains the direct sum of its three subgroups, each of the cardinality $\omega_1$. Therefore, the same is true for the original group $\Gamma$.

Notice that this lemma may also be proved independently of the general theory of Abelian groups. Namely, the required three subgroups of the group $\Gamma$ can be constructed by the method of transfinite recursion until $\omega_1$.

In the sequel we need two auxiliary notions. Let $\Gamma$ be an Abelian group, let $G$ be a fixed subgroup of $\Gamma$ and let $X$ be a subset of $\Gamma$. We say that the set $X$ is finite with respect to $G$ if for every element $x \in \Gamma$

the intersection $(x + G) \cap X$ is a finite subset of $\Gamma$. Analogously, we say that the set $X$ is countable with respect to $G$ if for every element $x \in \Gamma$ the intersection $(x + G) \cap X$ is a countable subset of $\Gamma$.

**Lemma 2.** *Let $\Gamma$ be an arbitrary Abelian group, let $G$ be a fixed uncountable subgroup of $\Gamma$ and let $X$ be a subset of $\Gamma$ finite with respect to $G$. Then $X$ is a $\Gamma$–negligible subset of $\Gamma$.*

The proof of Lemma 2 is not difficult. It can be carried out using the methods developed in the works [1] and [2]. Let us remark in connection with this lemma that if a subset $X$ of an Abelian group $\Gamma$ is countable with respect to some uncountable subgroup $G$ of $\Gamma$ then it is not true in general that $X$ is $\Gamma$-negligible in $\Gamma$. The corresponding counterexamples can be found in the work [1].

The next auxiliary result is due, in fact, to Sierpiński (see, for instance, a well known monograph [3] of Sierpiński). Notice only that in this monograph Sierpiński formulates and proves a beautiful geometric equivalent of the Continuum Hypothesis in terms of a partition of the three-dimensional Euclidean space into certain three subsets each of which is finite with respect to the corresponding axis of coordinates. But for our purpose we must reformulate Sierpiński's result mentioned above in terms of a similar partition of the direct sum of any three Abelian groups, each of the cardinality $\omega_1$. Hence, we do not need here the Continuum Hypothesis.

**Lemma 3.** *Let $\Gamma_1$, $\Gamma_2$ and $\Gamma_3$ be arbitrary Abelian groups, each of the cardinality $\omega_1$ and let $\Gamma$ be the direct sum of these groups. Then there exists a partition of $\Gamma$ into three sets $X$, $Y$ and $Z$ such that*

1) *the set $X$ is finite with respect to $\Gamma_1$;*
2) *the set $Y$ is finite with respect to $\Gamma_2$;*
3) *the set $Z$ is finite with respect to $\Gamma_3$.*

*In particular, all these three sets are $\Gamma$–negligible in the group $\Gamma$.*

We need also the following auxiliary proposition.

**Lemma 4.** *Let $(\Gamma, +)$ be an arbitrary Abelian group and let $G$ be a fixed subgroup of $\Gamma$. Let $G_0$ be a subgroup of the group $G$ and let $X$ be a subset of $G$ finite with respect to $G_0$. Suppose also that $H$ is a subset of the group $\Gamma$ which has one-element intersection with* *every $G$-orbit in $\Gamma$ (in other words, suppose that $H$ is a selector of the family of all $G$-orbits in $\Gamma$). Then the subset $H+X$ of the group $\Gamma$ is also finite with respect to the group $G_0$. In particular, if the group $G_0$ is uncountable then the set $H+X$ is $\Gamma$-negligible in $\Gamma$.*

Using the above lemmas it is not difficult to prove the following proposition.

**Proposition 1.** *Let $\Gamma$ be an arbitrary uncountable Abelian group. Then there exists a partition of this group into some three $\Gamma$-negligible sets. In particular, for every probability $\Gamma$-quasi-invariant measure $\mu$ defined on $\Gamma$ at least one of these three sets is non-measurable with respect to $\mu$ (in fact, at least two of these three sets are non-measurable with respect to $\mu$).*

By the same method a slightly more general result than Proposition 1 can be obtained. Namely, if $\Gamma$ is an Abelian group and $G$ is any uncountable subgroup of $\Gamma$ then there exists a partition of the group $\Gamma$ into three $G$-negligible subsets of $\Gamma$.

*Remark 1.* Using the same method we can prove a certain topological analogue of Proposition 1. Let $\Gamma$ be an arbitrary group. Let us consider the class $S(\Gamma)$ of all topologies $T$ defined on $\Gamma$ and satisfying the following relations:

a) $\Gamma$ is a second category space with respect to $T$;
b) the Suslin number $c(T)$ is equal to $\omega$, i.e. $T$ satisfies the countable chain condition;
c) all (left) translations of $\Gamma$ preserve the ideal of first category sets with respect to $T$ and the algebra of sets having the Baire property with respect to $T$.

Let $X$ be a subset of $\Gamma$. We say that this subset is $\Gamma$-negligible in the topological sense if

1) there exists a topology $T$ from the class $S(\Gamma)$ such that $X$ has the Baire property with respect to $T$;
2) for each topology $T'$ from the class $S(\Gamma)$, if $X$ has the Baire property with respect to $T'$ then $X$ is a first category set with respect to $T'$.

Now, a result analogous to Proposition 1 can be formulated in the following way: for every uncountable Abelian group $\Gamma$ there exists a partition of $\Gamma$ into some three $\Gamma$-negligible (in the topological sense) subsets of $\Gamma$. Hence, if $T$ is any topology from the class $S(\Gamma)$ then at least one of these three subsets does not have the Baire property with respect to $T$. Moreover, at least two of these three subsets do not have the Baire property with respect to $T$.

Let us formulate another proposition about the negligible sets which is sometimes useful if we want to extend Proposition 1 to the cases where the considered uncountable group $\Gamma$ is not necessarily commutative.

**Proposition 2.** *Let $\Gamma_1$ and $\Gamma_2$ be any two groups and let $f$ be any homomorphism from $\Gamma_1$ onto $\Gamma_2$. If a subset $Y$ of the group $\Gamma_2$ is $\Gamma_2$-negligible then the subset $X = f^{-1}(Y)$ of the group $\Gamma_1$ is $\Gamma_1$-negligible.*

*Remark 2.* Proposition 2 has a direct analogue for the so called absolutely non-measurable subsets of uncountable groups (in connection with this notion see [1], [2] and, especially, [4]). More exactly, let $\Gamma_1$ and $\Gamma_2$ be any two groups and let $f$ be any homomorphism from $\Gamma_1$ onto $\Gamma_2$. If a subset $Y$ of the group $\Gamma_2$ is absolutely $\Gamma_2$-non-measurable in $\Gamma_2$ then the subset $X = f^{-1}(Y)$ of the group $\Gamma_1$ is absolutely $\Gamma_1$-non-measurable in $\Gamma_1$. There are also some other interesting analogies between negligible sets and absolutely non-measurable sets in uncountable groups.

Let us return to Proposition 1. It shows, in particular, that for each uncountable Abelian group $\Gamma$ the class of all $\Gamma$-negligible subsets of $\Gamma$ is not closed even with respect to the finite unions. So, this class is not even a proper ideal of subsets of $\Gamma$. Therefore, we see that $\Gamma$-negligible sets are not, in fact, very small subsets of $\Gamma$.

The following definition describes a certain subclass of the class of all $\Gamma$-negligible sets. It turns out that this subclass is a proper ideal of subsets of a basic space $E$.

Let E be a non-empty basic space, let $\Gamma$ be a group of transformations of $E$ and let $X$ be a subset of $E$. We say that the set $X$ is absolutely $\Gamma$-negligible in $E$ if for every probability $\Gamma$-quasi-invariant measure $\mu$ defined on $E$ there exists a probability $\Gamma$-quasi-invariant measure $\lambda$ also defined on $E$ extending $\mu$ and such that $\lambda(X) = 0$.

It immediately follows from this definition that the class of all absolutely $\Gamma$-negligible sets is a proper ideal of subsets of the space $E$. So, we may conclude that no analogue of Proposition 1 can be proved for absolutely $\Gamma$-negligible sets. On the other hand, we shall see below that every uncountable Abelian group $\Gamma$ admits a countable partition into absolutely $\Gamma$-negligible sets.

We need an auxiliary proposition which yields a purely geometric characterization of absolutely negligible sets and plays an essential role during investigation of these sets.

**Lemma 5.** *Let $E$ be a non-empty basic space, let $\Gamma$ be a group of transformations of $E$ and let $X$ be a subset of $E$. Then the next two relations are equivalent:*

1) *$X$ is an absolutely $\Gamma$-negligible set in $E$;*
2) *for each countable family $\{g_i : i \in I\}$ of elements from the group $\Gamma$ there exists a countable family $\{h_j : j \in J\}$ of elements from $\Gamma$ such that*

$$\cap_j \ h_j(\cup_i \ g_i(X)) = \emptyset.$$

For the proof of Lemma 5 see [1] or [2]. Let us notice in connection with the result of this lemma that it would be interesting to obtain a similar purely geometric characterization of negligible sets.

Using Lemma 5 it is not difficult to prove the following

**Lemma 6.** *Let $(\Gamma, +)$ be an arbitrary Abelian group, let $G$ be a fixed subgroup of $\Gamma$ such that the cardinality of the family of all $G$-orbits in $\Gamma$ is less or equal to $\omega$. Let $H$ be any selector of the family of all $G$-orbits in $\Gamma$. If a subset $X$ of the group $G$ is absolutely $G$-negligible in $G$ then the subset $H + X$ of the group $\Gamma$ is absolutely $\Gamma$-negligible in $\Gamma$.*

Now, we can formulate the following result

**Proposition 3.** *Let $\Gamma$ be an arbitrary uncountable Abelian group. Then there exists a countable partition $\{X_i : i \in I\}$ of this group into absolutely $\Gamma$-negligible sets. In particular, for every probability $\Gamma$-quasi-invariant measure $\mu$ defined on $\Gamma$ there exists an index $i \in I$ (certainly, depending on $\mu$) such that the corresponding set $X_i$ is non-measurable with respect to $\mu$. Hence, the measure $\mu$ can be strictly* *extended to a probability $\Gamma$-quasi-invariant measure $\lambda$ also defined on $\Gamma$ and satisfying the equality $\lambda(X_i) = 0$.*

Let us make some remarks in connection with Proposition 3. In fact, the result of this proposition is contained in the work [1], while it is not formulated there. A short proof of Proposition 3 may be done, using the methods of [1], in the following way. According to the well known general theorems of the theory of Abelian groups (see, for instance, [5]), the uncountable Abelian group $\Gamma$ under our consideration can be represented as the union of an increasing (with respect to inclusion) countable family of subgroups $\{\Gamma_n : n \in \omega\}$ in such a way that each subgroup $\Gamma_n$ is the direct sum of cyclic groups. Now, only two cases are possible.

If for every natural index $n$ the cardinality of the family of all $\Gamma_n$-orbits in the group $\Gamma$ is strictly greater than $\omega$ then, applying Lemma 5, we immediately obtain that all sets $\Gamma_n$ are absolutely $\Gamma$-negligible in the group $\Gamma$. Using this fact it is easy to get the required countable partition of $\Gamma$ into absolutely $\Gamma$-negligible sets.

Suppose now that there exists a natural index $n$ such that the cardinality of the family of all $\Gamma_n$-orbits in the group $\Gamma$ is less or equal to $\omega$. Then it is obvious that the group $\Gamma_n$ is also uncountable and, moreover, it can be represented as the direct sum of two subgroups one of which has the cardinality $\omega_1$. Hence, according to the results of [1], there exists a countable partition of the group $\Gamma_n$ into absolutely $\Gamma_n$-negligible sets. From this fact, taking into account Lemma 6, we can conclude that there exists a countable partition of the group $\Gamma$ into absolutely $\Gamma$-negligible sets.

Therefore, in both cases we have the required result, and Proposition 3 is proved.

In connection with Proposition 1 and Proposition 3 the following two natural questions arise.

1. For what uncountable groups $\Gamma$ an analogue of Proposition 1 is true?
2. For what uncountable groups $\Gamma$ an analogue of Proposition 3 is true?

These questions are still open. Of course, the second question is more important from the point of view of the theory of quasi-invariant (in particular, invariant) measures. Notice that if the cardinality of the group $\Gamma$ is equal to $\omega_1$ then we have a direct analogue of Proposition 3 for this group. More generally, if $E$ is a basic space of the cardinality $\omega_1$ and $\Gamma$ is a transitive group of transformations of $E$ acting freely on $E$ then there exists a countable partition of $E$ into absolutely $\Gamma$-negligible sets (see [1] or [2]).

We also want to remark here that all the results above remain valid if we consider the class of non-zero $\sigma$-finite quasi-invariant measures instead of the class of probability quasi-invariant measures. Moreover, these results remain valid for the class of quasi-invariant (in particular, invariant) measures which satisfy the Suslin condition (i.e. the countable chain condition). Finally, these results show also that many facts of the theory of $\Gamma$-quasi-invariant (in particular, $\Gamma$-invariant) measures are connected only with the algebraic structure of the group $\Gamma$ of transformations of the space $E$.

At the end of this paper let us consider a topological application of Proposition 3. Let $\Gamma$ be an arbitrary group. Denote by the symbol $O(\Gamma)$ the class of all topologies $T$ defined on $\Gamma$ and satisfying the following three relations:

a) $T$ is a Baire space topology on $\Gamma$;
b) the Suslin number $c(T)$ is equal to $\omega$, i.e. $T$ satisfies the countable chain condition;
c) all (left) translations of the group $\Gamma$ preserve the ideal of first category sets with respect to $T$ and the algebra of sets having the Baire property with respect to $T$.

In particular, if $(\Gamma, T)$ is any $\sigma$-compact locally compact topological group then, of course, the topology $T$ belongs to the class $O(\Gamma)$.

We have the following result.

**Proposition 4.** *Let $\Gamma$ be an arbitrary uncountable Abelian group and let $T$ be any topology from the class $O(\Gamma)$. Then there exists a topology $T'$ in this class such that*

1) *$T'$ strictly extends $T$;*
2) *the ideal of first category sets with respect to $T$ is strictly contained in the ideal of first category sets with respect to $T'$.*

The proof of Proposition 4 can be obtained using the result of

Proposition 3. Indeed, we have a countable partition $\{X_i : i \in I\}$ of the group $\Gamma$ into absolutely $\Gamma$-negligible sets. Obviously, at least one set $X_i$ is not a first category subset of $\Gamma$ with respect to the original topology $T$. So, we can extend both the topology $T$ and the ideal of first category sets with respect to $T$ using the mentioned set $X_i$.

## References

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## MAŁE ZBIORY W NIEPRZELICZALNYCH GRUPACH ABELOWYCH

W pracy rozważa się pewne własności małych i absolutnie małych zbiorów w nieprzeliczalnych grupach abelowych.

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