ACTA UNIVERSITATIS LODZIENSIS
FOLIA MATHEMATICA 8, 1996

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REGULARITY AND COREGULARITY IN A CATEGORY WITH LOCALIZATION

To Professor Lech Włodarski on His 80th birthday

A category C together with a covariant functor T from C to the category Top of all topological spaces allows us to construct a category CT_0 of pairs (M, A) as objects, where A is a set of points of the set of all points of $T(M)$. Next, a covariant functor L from a subcategory of CT_0 to C is considered. In a category C equipped with covariant functors T and L satisfying some natural axioms of localization the concepts of regularity, weak regularity, coregularity and weak coregularity of morphisms of C is introduced and some categorial features of these concepts are established.

1. Category with localization

Any topological space X will be identified with its topology. X will be treated as a set of sets fulfilling two conditions: 1) $\bigcup Y \in X$ for $Y \subset X$, 2) $A \cap B \in X$ for $A, B \in X$. The set $\bigcup X$ of all points of X will be denoted by $\underline{X}$. For any $S \subset \underline{X}$ we adopt $X|S = \{A \cap S;\ A \in X\}$. Then we have a subspace $X|S$ of X with $\underline{X|S} = S$. We consider the category Top of all topological spaces together with all the morphisms

$$(1.0) \qquad (X, \varphi, Y),$$

where φ is a continuous function from X to Y. For any morphism t of form (1.0) and any $x \in \underline{X}$ the value $\varphi(x)$ will be also denoted by tx. For any $A \subset X$ the φ-image of A will be denoted by tA. If $tA \subset B \subset \underline{Y}$, then we have a morphism $(X|A, \ \ A \ni x \mapsto tx, \ \ Y|B)$, which will be denoted by t_A^B.

Let C be any category and T be a covariant functor from C to Top. For any object M of C we set $\underline{M} = \overline{T(M)}$. Then $\underline{M}$ is an object of the category Set. For any morphism f of C, i.e.

$$f : M \to N \tag{1.1}$$

and for any $x \in \underline{M}$ we set $fx = T(f)x$. Then we have a morphism $T_0(f)$ of Set of the form

$$(\underline{M}, \ \underline{M} \ni x \mapsto fx, \ \underline{N}),$$

where

$$T(f) = (T(M), \ \underline{M} \ni x \mapsto fx, \ T(N)).$$

Setting $T_0(M) = \underline{M}$ we get the functor T_0 from C to Set. For any $A \subset \underline{M}$ the set of all fx, where $x \in A$, will be denoted by fA.

We consider a category CT_0 meant as follows. All pairs (M, A), where $A \subset \underline{M}$, will be treated as objects of CT_0, while all the triplets

$$(A, f, B), \tag{1.2}$$

where $A \subset \underline{M}$, for morphism (1.1) with $fA \subset B \subset \underline{N}$, will be treated as a morphism of CT_0.

A morphism (B_1, g, E) will be treated as composable with (1.2) iff $B = B_1$ and $N = N_1$, where $g : N_1 \to P$. Then

$$(B, g, E) \cdot (A, f, B) = (A, g \cdot f, E).$$

We get in such a way a category CT_0. The functor T will be assumed to satisfy the following two conditions

(i) for any morphism (1.1) and any $g : M \to N$ the equality $T(f) = T(g)$ yields $f = g$;

(ii) for any objects M_1 and M_2 in C there exists their product (M, π), where $\pi = (\pi_1, \pi_2)$, in category C such that $T(M) = T(M_1) \times T(M_2)$ and

$$T(\pi_j) : T(M_1) \times T(M_2) \to T(M_j),$$

are the natural projections of the Cartesian product $T(M_1) \times T(M_2)$ onto $T(M_j)$ in category Top, $j = 1, 2$.

The above conditions (i) and (ii) yield that for any objects M_1 and M_2 of C we have the only product (M, π). We will set

$$M_1 \times M_2 = M, \qquad \mathrm{pr}_{jM_1 M_2} = \pi_j, \qquad j = 1, 2.$$

The full subcategory of CT_0 with the class of all objects of the form (M, A), where $A \in T(M)$ will be denoted by CT. Thus we have

$$(A, f, B) : (M, A) \to (N, B)$$

in the category CT iff (1.1), $A \in T(M)$, $B \in T(N)$ and $fA \subset B$.

Let us consider a covariant functor L from a subcategory CTL of CT_0 to the category C. We will assume that CT is a subcategory of CTL. For any object (M, A) and any morphism (1.2) of CTL we set

$$M_A = L(M, A) \quad \text{and} \quad f_A^B = L(A, f, B).$$

The functor L from CTL to C will be called a localization functor of T iff the following conditions (iii)–(vi) are satisfied.

(iii) for any morphism (1.2) of CTL we have

$$T(f_A^B) : T(M)|A \to T(N)|B \quad \text{and} \quad T(f_A^B) = T(f)_A^B;$$

(iv) for any morphism (1.2) of CTL, where (1.1) and any sets A', B' such that $(A', f_A^B B')$ is a morphism of CTL we have

$$(f_A^B)_{A'}^{B'}, \qquad M_{\underline{M}} = M \quad \text{and} \quad f_{\underline{M}}^{\underline{N}} = f;$$

(v) for any objects (M_1, A_1) and (M_2, A_2) of CTL we have

$$M_{1\ A_1} \times M_{2\ A_2} = (M_1 \times M_2)_A$$

and

$$\mathrm{pr}_{j\ M_1\ A_1 M_2\ A_2} = (\mathrm{pr}_{j\ M_1\ M_2})_A^{A_j}, \qquad j = 1, 2,$$

where $A = A_1 \times A_2$, and for any $a_1 \in A_1$ and $a_2 \in A_2$ there are morphisms

$$i_k : M_{k\ A_k} \to M_{1\ A_1} \times M_{2\ A_2}$$

such that $i_1 x = (x, a_2)$ for $x \in A_1$ and $i_2 x = (a_1, x)$ for $x \in A_2$;

(vi) if (M, A) and (M, A') are objects of CTL and $A' \subset A$, then (A', id_M, A) is an object of CTL, where $\mathrm{id}_M : M \to M$ is the identity morphism of the object M in C.

Remark 1. If CTL is a full subscategory of CT_0, then (vi) follows from the previous ones.

Remark 2. By (iii) we have, for any object (M, A) of CTL,

$$T(M_A) = T(M)|A.$$

A category C together with a covariant functor T from C to Top satisfying (i) and (ii), and with a localization functor L of T will be called a category with localization (c.l.).

2. Examples

We start with a trivial example

2.0. $C = \mathrm{Top}$, $T(X) = X$, $T(t) = t$, $L(A, t, B) = t_A^B$ for any object X, any morphism $t : X \to Y$ and any $A \subset \underline{X}$ and $B \subset \underline{Y}$ with $tA \subset B$.

2.1. Let k be any natural number or $k = \infty$. Let C be the category of all differential manifolds of class C^k together with all the C^k-mappings of differential manifolds as morphisms. For any C^k-mapping (1.1) we have the continuous mapping $T(f) = (T(M),\ \underline{M} \ni x \mapsto fx,\ T(N))$, where $T(M)$ and $T(N)$ denote the topology of the manifold M and N, respectively. For any $A \in T(M)$, $B \in T(N)$ and

(1.1) with $fA \subset B$ let $L(M, A)$ and $L(A, f, B)$ be the open submanifold M_A of M and the smooth mapping $f_A^B : M_A \to N_B$ from M_A into N_B induced by f. Similarly, for the category C of all analytical real (or complex) manifolds together with all the analytical mappings between them as well as for the category C of all differential Banach manifolds [4] together with all smooth mappings we define a functor T. We have the same situation in the case of so-called Aronszajn's subcartesian spaces [1] (see also [2]) as well as in the case of Banachian differentiable spaces [3] (see also [6]).

2.2. Let C be the category of all differential manifolds of class C^∞ together with all smooth mappings as morphisms. All the pairs (M, A) such that there exists an object P of C with the following conditions: 1) A is the set of all points of P, 2) the topology of P coincides with the one induced by the topology of M to the set A, 3) the identity mapping $\text{id} : P \to M$ is regular, i.e. the tangent mapping $T_p \text{id} : T_p P \to T_p M$ is a monomorphism. There is the only differential manifold P satisfying 1)–3). We denote this manifold by M_A. For any such pairs (M, A) and (N, B) and any smooth mapping (1.1) with $fA \subseteq B$ we have the induced mapping $f_A^B : M_A \to N_B$. Taking as $T(M)$ the topology of M and for any (1.1) $T(f) = (T(M), T(M) \ni x \mapsto fx, T(N))$, and $L(A, f, B) = f_A^B$ we get a (c.l.).

2.3. Let C be the category of all R. Sikorski's differential spaces [5] together with all the smooth mappings of differential spaces. All pairs M of the smooth mappings of differential spaces. All pairs M of the form $(\underline{M}, F(M))$, where $\underline{M}$ is a set and $F(M)$ is a set of real functions defined on $\underline{M}$ such that: 1) for any $a_0, \ldots, a_m \in F(M)$, where $m \in N$, and any C^∞-smooth function $C : \mathbb{R}^m \to \mathbb{R}$ the function $c(a_1(), \ldots, a_m()) \in F(M)$, 2) every function $b : \underline{M} \to \mathbb{R}$ such that for any $p \in \underline{M}$ there exists $U \in \text{top}\, M$, $p \in U$ (here $\text{top}\, M$ stands for the smallest of all topologies on $\underline{M}$ with continuous all the functions belonging to $F(M)$), and $a \in F(M)$ with $b|U = a|U$, belongs to $F(M)$, are treated as objects of C.

Morphisms in this category are all the triplets (M, φ, N), where

φ is a function with the domain $\underline{M}$ and the set of values in $\underline{N}$ such that for any $b \in F(N)$ we have $b \circ \varphi \in F(M)$. For any $A \subset \underline{M}$ let $F(M)_A$ be the set of all $b : A \to \mathbb{R}$ such that for any $p \in A$ there exists $U \in \operatorname{top} M$, $p \in U$, and $a \in F(M)$ with $b|A \cap U = a|A \cap U$. Setting $M_A = (A, F(M)_A)$ we get an R. Sikorski's differential space. We have $\operatorname{top} M_A = (\operatorname{top} M)|A$. Taking

$$(2.1) \qquad T(M) = \operatorname{top} M \quad \text{and} \quad T(f) = (T(M), \varphi, T(N))$$

we get a covariant functor from C to Top.

Next, setting

$$(2.2) \qquad L(M, A) = M_A \quad \text{and} \quad L(f) = (M_A, \varphi|A, N_B)$$

we get a localization functor of T such that $T = T_0$ and $CTL = CT_0$.

2.4. Let $K = \mathbb{R}$ or $K = \mathbb{C}$. For any set M of functions with values in K let $\underline{M} = \bigcup_{a \in M} D_a$, where D_a stands for the domain of the function a. Let $\operatorname{top} M$ be the smallest topology on $\underline{M}$ containing the set

$$\{a^{-1}B; \quad a \in M \quad \text{and } B \text{ is open in } \mathbb{R}\}.$$

For any $A \subset \underline{M}$ let M_A denote the set of all functions b with values in K such that for $p \in D_b$ there exist $U \in \operatorname{top} M$ and $a \in M$ with $p \in A \cap U \subset D_b$, $U \subset D_a$ and $b|A \cap U = a|A \cap U$. The set of all the functions $c(a_0, \dots, a_m)$, where $a_0, \dots, a_m \in M$, c is any function with values in K analytical on an open set D_c in K^m, $m \in \mathbb{N}$, is denoted by $\operatorname{an} M$. Here

$$D_{c(a_0,\dots,a_m)} = \{p;\ p \in D_{a_0} \cap \dots \cap D_{a_m} \quad \text{and} \quad (a_0(p), \dots, a_m(p)) \in D_c\}$$

and

$$c(a_0, \dots, a_m)(p) = c(a_0(p), \dots, a_m(p)) \quad \text{for} \quad p \in D_{c(a_0,\dots,a_m)}.$$

A set M of functions with values in K satisfying the equalities: $M = M_{\underline{M}} = \operatorname{an} M$ is said to be a general differential space (g.d.s) [8] and [7].

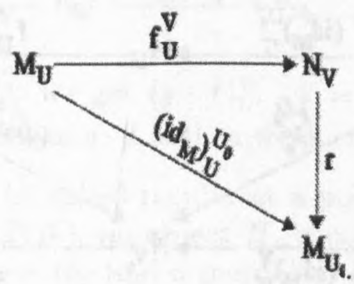
All g.d.s. we treat as objects of a category C. All triplets (M, φ, N), where φ is a function with the domain $\underline{M}$ and values in $\underline{N}$ and such that $b \circ \varphi \in M$ for $b \in N$ are treated as morphisms of C. Here $D_{b\circ\varphi} = \varphi^{-1}D_b$. The composition of morphisms is defined in the usual way. Setting (2.1) we get a covariant functor from C to Top. Defining L on the same formal way as in Example 2.3 by formulae (2.2) we get a localization functor of T. Here $T = T_0$ and $CTL = CT_0$.

3. Weak regularity, regularity, weak coregularity and coregularity

Let C, T, L be a c.l. A morphism (1.1) of C will be called weak regular at the point $p \in \underline{M}$ iff there exist U, $U_0 \in T(M)$, $V \in T(N)$ and a morphism

$$r : N_V \to M_{U_0}$$

such that $p \in U \subset U_0$, $fU \subset V$ and we have commutative diagram



Morphism (1.1) of C will be called weak coregular at the point $p \in \underline{M}$ iff there exist $U \in T(M)$, $V, V_0 \in T(N)$, $V_0 \subset V$ and a morphism $s : N_{V_0} \to M_U$ such that $p \in U$, $fU \subset V$, $fp \in V$, $sfp = p$ and $f_U^V \cdot s = (\mathrm{id}_N)_{V_0}^V$.

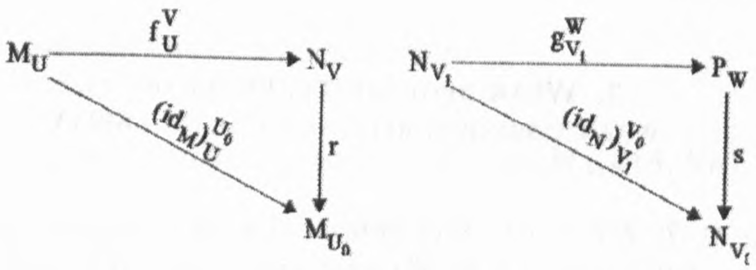
Morphism (1.1) weak regular (weak coregular) (cf. [9]) at every point $p \in \underline{M}$ is said to be weak regular (weak coregular).

3.1. Proposition. *If (1.1) and*

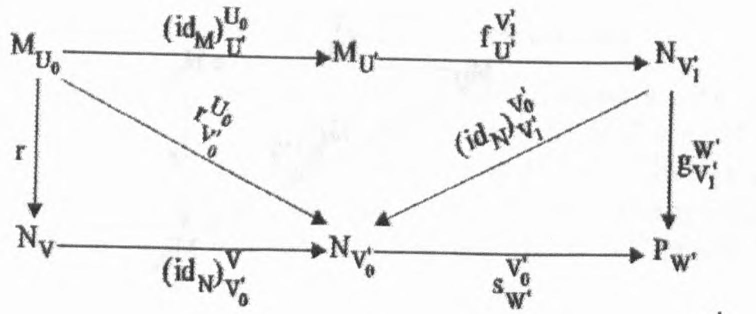
(3.1) $$g : N \to P$$

are weak regular (weak coregular), then $g \cdot f : M \to P$ is.

Proof. Let $p \in \underline{M}$. Weak regularity of (1.1) at p as well as of (3.1) at fp yield the existence of $U, U_0 \in T(M)$, $V, V_0, V_1 \in T(N)$, $W \in T(P)$ and morphisms r and s such that $p \in U \subset U_0$, $fU \subset V$, $fp \in V_1 \subset V_0$, $gV_1 \subset V_0$ and the diagrams

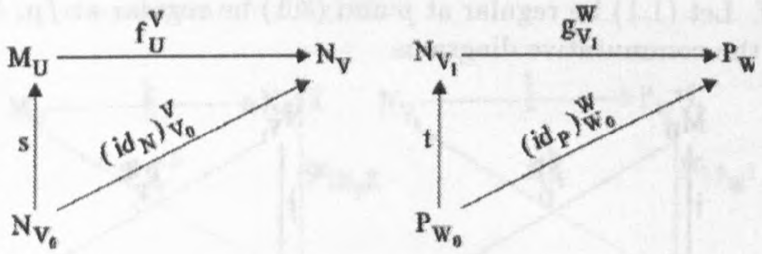


are commutative. Setting $V_0' = V_0 \cap V$ and $U' = f^{-1}V$ (= the set of all $x \in \underline{M}$ with $fx \in V$) we get the commutative diagram



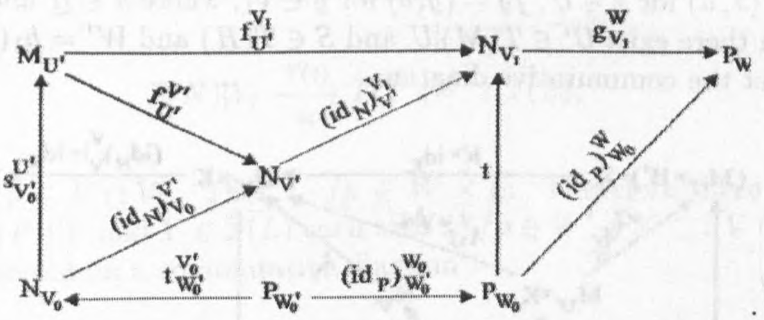
Taking $t = r^{U_0}_{V_0'} \cdot s^{V_0'}_{W'}$ we get $t \cdot (g \cdot f)^{W'}_{U'} = r^{U_0}_{V_0} \cdot s^{V_0'}_{W'} \cdot g^{W'}_{V_1'} \cdot f^{V_1'}_{U'} = r^{U_0}_{V_0} \cdot (s \cdot g^{W}_{V_1})^{V_0'}_{V_1'} = r^{U_0}_{V_0'} \cdot (\mathrm{id}_N)^{V_0'}_{U'} = r^{U_0}_{V_1'} \cdot f^{V_1'}_{U'} = (r \cdot f^{V}_{U})^{U_0}_{U'} = (\mathrm{id}_M)^{U_0}_{U'}$ and, of course, $p \in U$. Thus $g \cdot f$ is regular at p.

Similarly, weak coregularity of morphisms (1.1) and (3.1) at any $p \in \underline{M}$ and at fp, respectively, yields the existence of $U \in T(M)$, $V_0, V, V_1 \in T(N)$, $W_0, W \in T(P)$ and morphisms s and t such that $p \in U$, $fp \in V_0$, $sq = p$, $q = fp \in V_1$, $gq \in W_0$, $tgq = q$, $V_0 \subset V$, $fU \subset V$, $gV_1 \subset W$ and the diagrams



are commutative.

Assuming $V' = V_1 \cap V$, $U' = f^{-1}V'$, $V_0' = s^{-1}U'$, $W_0' = f^{-1}V_0$ we get the commutative diagram



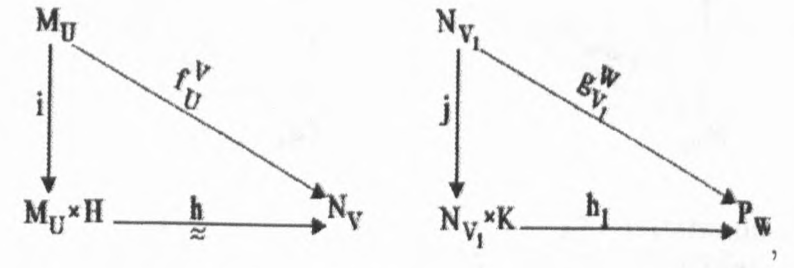
Setting $z = s^{u'}_{V_0'} \cdot t^{V_0'}_{W_0'}$ we get $(g \cdot f)^W_{U'} \cdot z = (\mathrm{id}_p)^W_{W'}$, $p \in W_0'$, $z \cdot g \cdot fp = p$. The morphism $g \cdot f$ is then weak coregular at p.

Morphism (1.1) will be called regular at a point $p \in \underline{M}$ iff there exist $U \in T(M)$, $V \in T(N)$, an object H, a point $a \in \underline{H}$, an isomorphism $h : M_U \times H \to N_V$ and a morphism $i : M_U \to M_U \times H$ such that $p \in U$, $ix = (x, a)$ for $x \in U$, $fU \subset V$ and $h \cdot i = f^V_U$. The morphism being regular at each point will be called regular.

Morphism (1.1) will be called coregular at a point p iff there exist $U \in T(M)$, $V \in T(N)$, an object K and an isomorphism $k : M_U \to N_V \times K$ such that $p \in U$, $fU \subset V$ and $\mathrm{pr}_{1\ N_V K} \cdot k = f^V_U$, $\mathrm{pr}_{1\ N_V K} : N_V \times K \to N_V$. The morphism coregular at every point is said to be coregular.

3.2. Proposition. *Every regular (coregular) morphism at a point is weak regular (weak coregular) at this point. The composition of regular (coregular) morphisms is regular (coregular).*

Proof. Let (1.1) be regular at p and (3.1) be regular at fp. We have then the commutative diagrams



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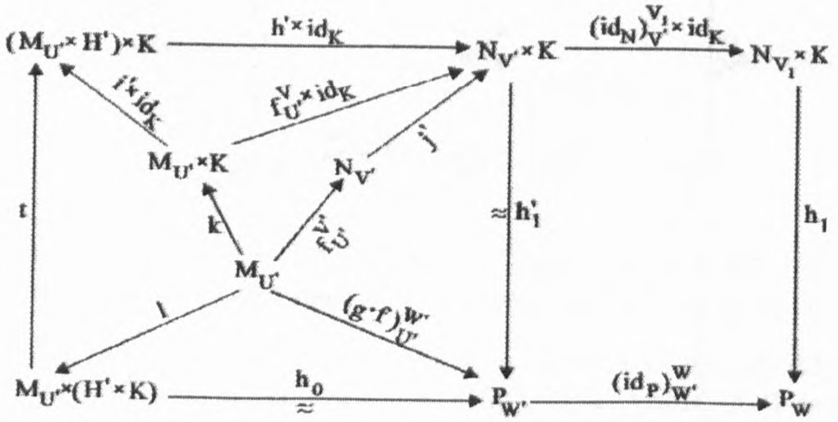
where $p \in U \in T(M)$, $fU \subset V$, $fp \in V_1 \in T(N)$, $gV_1 \subset W$, $ix = (x, a)$ for $x \in U$, $jy = (y, b)$ for $y \in V_1$, where $a \in \underline{H}$ and $b \in \underline{K}$. Then there exist $U' \in T(M)|U$ and $S \in T(H)$ and $W' = h_1(V' \times \underline{K})$ we get the commutative diagram



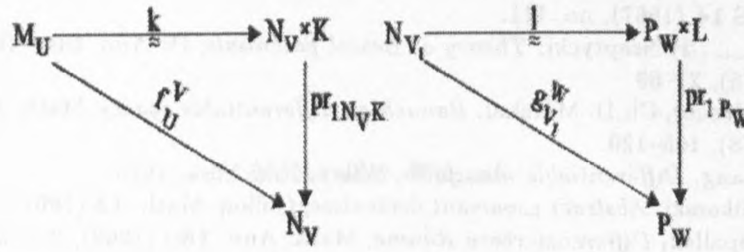
where $i' = i_{U'}^{U' \times S}$, $h' = h_{U' \times S}^{V'}$, $h_1 = h_1{}^{W'}_{V' \times \underline{K}}$, $j' = j_{V'}^{V' \times \underline{K}}$,

$$t : M_{U'} \times (H' \times K) \underset{\approx}{\longrightarrow} (M_{U'} \times H') \times K$$

is the canonical isomorphism of Cartesian products in the category C, $l : M_U \to M_{U'} \times (H' \times K)$ and $k : M_{U'} \to M_{U'} \times K$, $lx = (x, (a, b))$ and $kx = (x, b)$ for $x \in U'$. In particular, $(g \cdot f)_{U'}^{W'} = h_0 \cdot l$. The morphism $g \cdot f : M \to P$ is then regular at p.

To prove coregularity of the composition of coregular morphisms let us assume that (1.1) is regular at p and (3.1) is coregular at fp.

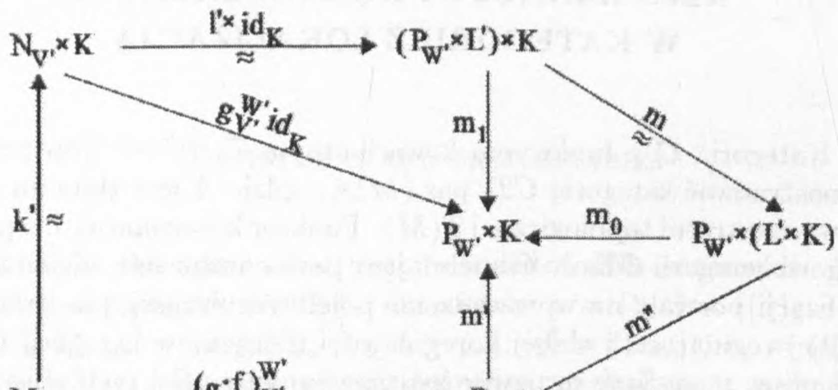
Then we have commutative diagrams



where $p \in U \in T(M)$, $fU \subset V \in T(N)$, $fp \in V_1 \in T(N)$, $gV_1 \subset W \in T(P)$. We have a homeomorphism

$$T(N)|V_1 \xrightarrow[\approx]{T(t)} T(P)|W \times T(L),$$

where $fp \in V \cap V_1$. Then $l \cdot fp \in W \times \underline{L}$. Therefore there exist $W' \in T(P)|W$ and $Y \in T(L)$ such that $l \cdot fp \in W' \times Y \subset l(V \cap V_1)$. Hence we obtain a commutative diagram



where $V' = l^{-1}(W' \times Y)$, $L' = L_Y$, $U' = k^{-1}(V' \times \underline{K})$, $m : (P_{W'} \times L') \times K \to P_{W'} \times (L' \times K)$ is the canonical isomorphism of Cartesian products, $m_1 = \mathrm{pr}_{1\ P_{W'}L'} \times \mathrm{id}_{K'}$, $m' = \mathrm{pr}_{1\ P_{W'}K}$ and $m'' = \mathrm{pr}_{1\ P_{W'}(L' \times K)}$.

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REGULARNOŚĆ I KOREGULARNOŚĆ W KATEGORII Z LOKALIZACJĄ

Kategoria C z funktorem kowariantnym $T : C \to \text{Top}$ pozwala skonstruować kategorię CT_0 par (M, A), gdzie A jest zbiorem punktów przestrzeni topologicznej $T(M)$. Funktor kowariantny L z podkategorii kategorii CT_0 do C spełniający pewne naturalne aksjomaty lokalizacji pozwala na wprowadzenie pojęć: regularości, koregularości, słabej regularości i słabej koregularości morfizmów kategorii C. W tej pracy omówione są pewne kategoryjne własności tych pojęć.

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