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ON APPROXIMATIONS OF FUNCTIONS ON METRIC SPACES

To Professor Lech Wlodarski on His 80th birthday

Let $\Phi : X \to Y$ be a linear family of Lipschitz function. We assume that the family Φ satisfies additional conditions. Under these assumptions we show the following result:

Let $\phi_x \in \Phi$ be such that for all $x, y \in X$

$$\|[\phi_x(y) - \phi_x(x)] - [f(y) - f(x)]\|_Y \leq K(d_X(x,y))^\alpha.$$

Then ϕ_x is uniquely determined up to a constant and it satisfies Hölder condition with exponent $\alpha - 1$ with respect to x in the Lipschitz norm $\|.\|_L$.

Since optimization in metric spaces, the convex analysis over metric spaces was developed (see [2]-[7]). In this paper we shall extend on a metric space the following classical theorem.

Theorem 1. *Let $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$, be Banach spaces. Let $F(x)$ be a differentiable mapping of an open set $U \subset X$ into Y. The differential $\partial F|_x$ as a function of x satisfies a Hólder condition with an exponent $0 < \alpha \leq 1$ and with constant $K > 0$ if and only if for each $x, y \in U$*

$$(1) \qquad \|[F(y) - F(x)] - \partial F_x(y - x)\|_Y \leq K\|y - x\|_X^{1+\alpha}.$$

The our extension concern the case when (X, d_X) is a metric space and $(Y, \|\cdot\|_Y)$ as before is a Banach space. Let Φ be a a family of mappings of an open set $U \subset X$ into Y. Let $F(x)$ be a mapping of the open set $U \subset X$ into Y. We say that a mapping $\phi \in \Phi$ is a *Φ-gradient* at a point x of $F(x)$ if for each $\varepsilon > 0$ there is a neighbourhood V of x such that for all $y \in V$

$$\|[F(y) - F(x)] - \phi(y) - \phi(x)\|_Y \leq \varepsilon d_X(y, x). \tag{2}$$

We say that a mapping $F(x)$ mapping of the open set $U \subset X$ into Y is *Φ-differentiable* at a point x if for each x there is a Φ-gradient ϕ_x of the $F(x)$ at the point x. Observe that under such general formulation this Φ-gradient need not to be unique.

When we want to extend Theorem 1, we need to determine something which play a role of a norm of operator. Observe that in the case of linear operators the norms in nothing else as the Lipschitz constant.

Let (X, d_X) be a metric space. Let $(Y, \|\cdot\|_Y)$ be a Banach space. Let Φ be a linear class of Lipschitzian mapping of X with values in Y. We define on Φ a quasinorm

$$\|\phi\|_L = \sup_{x_1, x_2 \in X,\ x_1 \neq x_2} \frac{|\phi(x_1) - \phi(x_2)|}{d_X(x_1, x_2)}. \tag{3}$$

Observe that if $\|\phi_1 - \phi_2\|_L = 0$, then the difference of ϕ_1 and ϕ_2 is a constant function, i.e. $\phi_1(x) = \phi_2(x) + c$, where $c \in Y$. Thus we consider the quotient space $\tilde{\Phi} = \Phi / R$. The quasinorm $\|\phi\|_L$ induces the norm in the space $\tilde{\Phi}$. Since it will not lead to misunderstanding this norm we shall denote also $\|\phi\|_L$.

Theorem 2. *Let (X, d_X) be a metric space and let $(Y, \|\cdot\|_Y)$ be a Banach space. Let Φ denote a linear class of Lipschitzian functions defined on X with values in Y, such that for each $\phi \in \Phi$, $x \in X$, $t > 0$, $\delta > 0$, $\varepsilon > 0$ there is $y \in X$ such that*

$$|d_X(x, y) - t| < \delta t \tag{4}$$

and

$$(5) \qquad |\frac{\|\phi(x)-\phi(y)\|_Y}{d_X(x,y)} - \|\phi\|_L| < \varepsilon.$$

Let $F(x) : X \to Y$ be a Φ-differentiable function. Let ϕ_x be a Φ-gradient of the function $f(x)$ at a point x. Suppose that for all $x, y \in X$

$$(6) \qquad \|[\phi_x(y) - \phi_x(x)] - [f(y) - f(x)]\|_Y \leq \gamma(d_X(x,y)),$$

where the real valued function $\gamma(t)$ defined for $0 \leq t$ is independent on x. Let $\frac{\gamma(t)}{t}$ tends to 0 as t tends to 0. Then ϕ_x is uniquely determined up to the constant and

$$(7) \qquad \|\phi_x - \phi_Y\|_L \leq \omega(d_X(x,y)),$$

where $\omega(t) = \frac{\gamma(2t)+2\gamma(t)}{t}$.

Proof. Let x_0 be a fixed point in X. Let ϕ_{x_0} be a Φ-gradient of the function $f(x)$ at x_0. Now we shall use the fact that the class Φ is linear. Let $\tilde{f}(x) = f(x) - \phi_{x_0}(x)$. Observe that $\psi \in \Phi$ is a Φ-gradient of the function $\tilde{f}(x)$ at x_0 if and only if $\psi + \phi_{x_0}$ is a Φ-gradient of the function $f(x)$ at x_0. Thus we can assume without loss of generality that 0 is a Φ-gradient of the function $f(x)$ at x_0 and

$$(8) \qquad \|f(x) - f(x_0)\|_Y \leq \gamma(d_X(x,x_0)).$$

Now we shall show that 0 is a unique up to a constant Φ-gradient of the function $f(x)$ at x_0.

Indeed, let $\phi \in \Phi$ be an arbitrary Φ-gradient of the function $f(x)$ at x_0. Since $\frac{\gamma(t)}{t}$ tends to 0 as t tends to 0, by (8) for each $\varepsilon > 0$ there is a $t > 0$ such that $d_X(x,x_0) < t$ implies

$$(9) \qquad \|\phi(x) - \phi(x_0)\|_Y \leq \varepsilon d_X(x,x_0).$$

Thus by (5) and (9)

$$\|\phi\|_L \leq 2\varepsilon.$$

The arbitrariness of ε implies that $\|\phi\|_L = 0$. It show the uniqueness up to a constant of the Φ-gradient.

Let x_0 be an arbitrary point in X. Now we shall show (7). Similarly as before, without loss of generality we may assume that 0 is the Φ-gradient of the function $f(x)$ at x_0. Let x be another arbitrary point in X. We denote $d_X(x,x_0)$ by t, $t = d_X(x,x_0)$. Let ϕ_x denote the Φ-gradient of the function $f(x)$ at the point x. By our assumptions (4) for each $\delta > 0$, $\varepsilon > 0$ there is $y \in X$ such that

$$|d_X(x,y) - t| < \delta \tag{4}$$

and

$$\left|\frac{\|\phi_x(x) - \phi_x(y)\|_Y}{d_X(x,y)} - \|\phi_x\|_L\right| < \varepsilon. \tag{5}$$

Thus by (6) we have

$$\left|\frac{\|f(x) - f(y)\|_Y}{d_X(x,y)} - \|\phi_x\|_L\right| < \frac{\gamma(d_X(x,y))}{d_X(x,y)} + \varepsilon.$$

Therefore

$$\begin{aligned} \|\phi_x\|_L &\le \frac{\|f(y) - f(x)\|_Y}{d_X(x,y)} + \frac{\gamma(d_X(x,y))}{d_X(x,y)} + \varepsilon \\ &\le \frac{\|f(y)\|_Y}{d_X(x,y)} + \frac{\|f(x)\|_Y}{d_X(x,y)} + \frac{\gamma(d_X(x,y))}{d_X(x,y)} + \varepsilon. \end{aligned} \tag{10}$$

Recalling (4), we have

$$d_X(x,x_0) - \delta \le d_X(x,y) \le d_X(x,x_0) + \delta. \tag{11}$$

Thus

$$d_X(x_0,y) \le d_X(x,x_0) + d_X(x,y) \le 2d_X(x,x_0) + \delta. \tag{12}$$

Since 0 is a Φ-gradient of the function $f(x)$ at the point x_0, we obtain by (4) that

$$\|f(x)\|_Y \le \gamma(d_X(x,x_0))$$

and

$$\|f(y)\|_Y \leq \gamma(2d_X(x,x_0)+\delta)$$

Combining this estimation with (10) we obtain

$$\begin{aligned}\|\phi_x\|_L &\leq \frac{\gamma(2\,d_X(x,x_0)+\delta)}{d_X(x,x_0)-\delta} + \frac{\gamma(d_X(x,x_0))}{d_X(x,x_0)-\delta} + \frac{\gamma(d_X(x,y))}{d_X(x,y)} + \varepsilon \\ &\leq \frac{\gamma(2d_X(x,x_0)+\delta)}{d_X(x,x_0)-\delta} + 2\,\frac{\gamma(d_X(x,x_0)+\delta)}{d_X(x,x_0)-\delta} + \varepsilon.\end{aligned}$$

The arbitrariness of δ and ε finish the proof.

As an obvious consequence we obtain

Theorem 3. *Let (X, d_X) be a metric space and let $(Y, \|\cdot\|_Y)$ be a Banach space. Let Φ denote a linear class of Lipschitzian functions defined on X with values in Y, such that for each $\phi \in \Phi$, $x \in X$, $t > 0$, $\delta > 0$, $\varepsilon > 0$ there is $y \in X$ such that*

$$|d_X(x,y) - t| < \delta t \tag{4}$$

and

$$\left| \frac{\|\phi(x) - \phi(y)\|_Y}{d_X(x,y)} - \|\phi\|_L \right| < \varepsilon. \tag{5}$$

Let $f(x) : X \to Y$ be a Φ-differentiable function. Let ϕ_x be a Φ-gradient of the function $f(x)$ at a point x. Suppose that for all $x, y \in X$

$$\|[\phi_x(y) - \phi_x(x)] - [f(y) - f(x)]\|_Y \leq K(d_X(x,y))^\alpha, \tag{14}$$

where the constant $K > 0$ and the exponent α, $1 < \alpha \leq 2$, are independent on x.

Then ϕ_x is uniquely determined up to a constant and it satisfies Hölder condition with exponent $\alpha - 1$ with respect to x in the norm $\|.\|_L$. In particular case when $\alpha = 2$, ϕ_x as a function of x satisfies Lipschitz condition in the Lipschitz norm.

We say that a metric space (X, d_X) is *K-convex space* (see [8]), $K \geq 1$, if for each $x, y \in X$ and each $\alpha > 0$, there are elements

$x = x_0, x_1, \ldots, x_n = y$ such that $d_X(x_i, x_{i-1}) < \alpha$, $i = 1, 2, \ldots, n$ and

$$\sum_{i=1}^{n} d_X(x_i, x_{i-1}) \leq K d_X(x, y). \tag{15}$$

For $K = 1$, K-convex sets was firstly investigated by Menger [1] in 1928. The investigations are intensively developed till today (see for example [9]).

Let a metric space (X, d_X) be given. By a *curve* in X we shall understand a homeomorphic image L of the interval $[0,1]$, i.e. the function $x(t)$, $0 \leq t \leq 1$ defined on interval $[0,1]$ with values in X such that $x(t) = x(t')$ implies $t = t'$. The point $x(0)$ is called the *beginning of the curve*, the point $x(1)$ is called the *end of the curve*. By the length of a curve L we mean $l(L) = \sup\{\sum_{i=1}^{n} d_X(x(t_i), x(t_{i-1})) : 0 = t_0 < t_1 < \cdots < t_n = 1\}$.

We say that a metric space (X, d_X) is *arc connected* if for arbitrary $x_0, y \in X$ there is a function $x(t)$, $0 \leq t \leq 1$ defined on interval $[0,1]$ with values in X such that $x(0) = x_0$, $x(1) = y$ and the length of the line $L = \{x(t)\}$, $0 \leq t \leq 1$ can be estimated as follows $l(L) \leq K d_X(x_0, y)$.

If a metric space (X, d_X) is arc connected with a constant $K > 0$, then it is K-convex. The converse is not true. For example the set Q of all rational numbers with the standard metric is K-convex, but it is not arc connected with any constant $K \geq 1$. In the example the space X is not connected. However it is possible to construct a complete K-convex metric space (X, d_X), which is not arc connected with any constant $K \geq 1$. We want to mention, that a complete 1-convex metric space (X, d_X) is always arc connected with a constant 1.

As a consequence of Theorems 2 and 3 and the notion of arc connected spaces we obtain

Corollary 4. *Let (X, d_X) be an arc connected with a constant K metric space and let $(Y, \|\cdot\|_Y)$ be a Banach space. Let Φ denote a linear class of Lipschitzian functions defined on X with values in Y, such that for each $\phi \in \Phi$, $x \in X$, $t > 0$, $\delta > 0$, $\varepsilon > 0$ there is $y \in X$*

such that

$$|d_X(x,y) - t| < \delta t \tag{4}$$

and

$$\left| \frac{\|\phi(x) - \phi(y)\|_Y}{d_X(x,y)} - \|\phi\|_L \right| < \varepsilon. \tag{5}$$

Let $f(x) : X \to Y$ be a Φ-differentiable function. Let ϕ_x be a Φ-gradient of the function $f(x)$ at a point x. Suppose that for all $x, y \in X$

$$\|[\phi_x(y) - \phi_x(x)] - [f(y) - f(x)]\|_Y \leq \gamma(d_X(x,y)), \tag{6}$$

where the real valued function $\gamma(t)$ defined for $0 \leq t$ is independent on x. Let $\frac{\gamma(t)}{t^2}$ tends to 0. Then $f(x) = \phi(x) + c$, where $\phi \in \Phi$ and $c \in R$.

Proof. Since $\frac{\gamma(t)}{t^2}$ tends to 0, $\omega_1(t) = \frac{\gamma(2t)+2\gamma(t)}{t^2}$ tends to 0, too. Thus for each $\eta > 0$ there is $\alpha > 0$ such that $t < \alpha$ implies that $\omega_1(t) < \eta$. Therefore $\omega(t) < \eta t$.

Since X is arc connected with a constant K, it is K-convex. Thus there are elements $x = x_0, x_1, \ldots, x_n = y$ such that $d_X(x_i, x_{i-1}) < \alpha$, $i = 1, 2, \ldots, n$ and

$$\sum_{i=1}^{n} d_X(x_i, x_{i-1}) \leq K d_X(x,y). \tag{15}$$

By formula (7)

$$\|\phi_{x_i} - \phi_{x_{i-1}}\|_L \leq \omega(d_X(x_i, x_{i-1})) \leq \eta d_X(x_i, x_{i-1}), \tag{16}$$

for $i = 1, 2, \ldots, n$. Thus by the triangle inequality and by (15)

$$\begin{aligned} \|\phi_x - \phi_y\|_L = \|\phi_{x_0} - \phi_{x_n}\|_L &\leq \sum_{i=1}^{n} \|\phi_{x_i} - \phi_{x_{i-1}}\|_L \\ &\leq \eta \sum_{i=1}^{n} d_X(x_i, x_{i-1}) \leq K\eta d_X(x,y). \end{aligned} \tag{17}$$

The arbitrariness of η implies that

$$\|\phi_x - \phi_y\|_L = 0 \tag{18}$$

for arbitrary $x, y \in X$. Thus $\phi = \phi_x$ is a Φ-gradient of the function $f(x)$ at each point x. Take arbitrary $\hat{x}, y \in X$. Since the space X is arc connected with a constant K, there is a curve L with the beginning at $\hat{x}$ and end at y such that the length of L is not greater then $Kd_X(\hat{x}, y)$. Take arbitrary $\varepsilon > 0$ and arbitrary $x = x(t) \in L$. Then there is δ_t such that for z such that $d_X(x, z) < \delta_t$

$$\|[\phi(z) - \phi(x)] - [f(z) - f(x)]\|_Y < \varepsilon d_X(x, z). \tag{19}$$

Using the fact that L is compact we obtain that there are points $\hat{x} = x_0, x_1, \ldots, x_n = y$ such that

$$\sum_{i=1}^{n} d_X(x_i, x_{i-1}) \leq K d_X(\hat{x}, y) \tag{20}$$

and

$$\|[\phi(x_i) - \phi(x_{i-1})] - [f(x_i) - f(x_{i-1})]\|_Y \leq \varepsilon d_X(x_i, x_{i-1}), \tag{21}$$

for $i = 1, 2, ..., n$. Thus by the triangle inequality and by (20)

$$\|[\phi(\hat{x}) - \phi(y)] - [f(\hat{x}) - f(y)]\|_Y \leq \varepsilon d_X(\hat{x}, y), \tag{22}$$

The arbitrariness of ε implies that

$$[\phi(\hat{x}) - \phi(y)] - [f(\hat{x}) - f(y)] \tag{23}$$

and the arbitrariness of $\hat{x}, y$ implies that $f(x) = \phi(x) + c$.

Observe that in particular case when $\gamma(t) = t^{\alpha}$, if $\alpha > 2$ Corollary 4 holds.

We do not know is Corollary 4 true without assumption that the metric space X is not arc connected with constant K?

References

[1] K. Menger, *Untersuchen über allgemeine Metrik I – III*, Math. Ann. **100** (1928), 75–163.

[2] S. Rolewicz, *On Asplund inequalities for Lipschitz functions*, Arch. der Mathematik **61** (1993), 484–488.

[3] ———, *On extension of Mazur theorem on Lipschitz functions*, Arch. der Mathematik **63** (1994), 535–540.

[4] ———, *On a globalization property*, Appl. Math. **22** (1993), 69–73.

[5] ———, *Convex analysis without linerity*, Control and Cybernetics **23** (1994), 247–256.

[6] ———, *On subdifferentials on non-convex sets*, Different Aspects of Differentiablity, Diss.Math. (D. Przeworska-Rolewicz **340**, ed.), 1995, pp. 301–308.

[7] ———, *On Φ-differentiability of functions over metric spaces*, (submitted), Topological Methods of Nonlinear Analysis.

[8] R. Rudnicki, *Asymptotic properties of the iterates of positive operators on* $C(X)$, Bull. Pol. Acad. Sc. Math. **34** (1986), 181–187.

[9] V.P. Soltan, *Introduction in Axiomatic Theory of Convexity*, in Russian, Kishiniev, 1984.

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O APROKSYMACJI FUNKCJI W PRZESTRZENIACH METRYCZNYCH

Niech $\Phi : X \to Y$ będzie liniową rodziną funkcji Lipschitzowskich. Załóżmy, że rodzina Φ spełnia pewne dodatkowe warunki. Pod tymi założeniami pokazujemy następujące twierdzenie:

Twierdzenie. *Niech $\phi_x \in \Phi$ będzie takie, że dla wszystkich $x, y \in X$*

$$\|[\phi_x(y) - \phi_x(x)] - [f(y) - f(x)]\|_Y \leq K(d_X(x,y))^\alpha.$$

Wtedy ϕ_x jest jednoznacznie określona z dokładnością do stałej i *spełnia warunek Höldera z wykładnikiem* $\alpha - 1$ *ze względu na* x *w normie Lipschitzowskiej* $\|.\|_L$.

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