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ON SUMMABILITY METHODS DEFINED BY SEQUENTIAL MATRIX METHODS AND THOSE DEFINED BY THE ITERATION PRODUCTS OF MATRIX TRANSFORMATIONS

To Professor Lech Włodarski on His 80th birthday

We introduce sequential matrix methods, called shortly SM-methods, and show that they are equivalent to the well-known methods defined by iteration products of matrix transformations, being rather more complicated for investigations than SM-methods. Our main goal is to present result on the b-perfectness and the perfectness of regular SM-methods which can frequently be reformulated for iteration products of matrix transformations.

CONTENTS

0. Preliminaries

This paper is meant as a summary based mainly on doctoral thesis [7] and takes into account some modifications and supplements. We first introduce sequential matrix methods, called shortly SM-methods and consider these methods and those defined by the iteration products of matrix transformations [8], called IPM-methods as well. It turns out that the latter ones are equivalent to some SM-methods, and vice versa (Section 1.1). This implies that both the kinds of methods have global properties in common, that is, properties depending only on limit functionals. In general, the investigation of IPM-methods is rather more complicated than that of SM-methods. Therefore, many global properties of IPM-methods can be drawn from those of SM-methods.

We pay attention to the assignment which sends every IPM-method $\Theta[\Lambda]$ to the equivalent SM-method $\Theta\#\Lambda$. This assignment allows us to transfer the global properties of IPM-methods formulated in terms of their defining matrices to those of SM-methods formulated in terms of their defining (1,2)-indexed matrices. In particular, this concerns the regularity and the almost regularity (Sections 1.2 and 1.3). Moreover, in this way one can describe the topology in the full convergence field of an IPM-method (Section 3.1).

Our main goal is to present results on the b-perfectness and the perfectness of regular SM-methods (Chapters 2 and 3). These results can be reformulated for IPM-methods because the classes of IPM-methods and SM-methods are equivalent. From the viewpoint of global properties, the class $\mathcal{M}^2$ of regular SM-methods can be regarded as that containing the class $\mathcal{M}$ of regular matrix methods. In the case of the b-perfectness, we obtain the result stating that for any $A \in \mathcal{M}^2$ the following conditions are equivalent: A is b-perfect in $\mathcal{M}^2$, A is b-perfect in $\mathcal{M}$, and A is b-equivalent to a matrix method (Propositions 2.1.3 and 2.5.1), which implies that every matrix method is b-perfect in $\mathcal{M}^2$. In the case of the perfectness, we can distinguish the class $\mathcal{M}^2_r$ of regular reducible SM-methods which has properties similar to those of regular SM-methods relative to the b-perfectness (Corollary 3.4.4). In general, the perfectness for an SM-method can be investigated by using the corresponding topology in the full convergence field of such a method (Section 3.1). This topology generalizes that for a matrix method (see [9]). By applying it we get the following characterization: a regular SM-method A is perfect in $\mathcal{M}^2$ if and only if the set T_c is dense in the full convergence field of A (Theorem 3.2.2). In particular, any matrix method perfect in $\mathcal{M}$ is perfect in $\mathcal{M}^2$. The above-mentioned results on the b-perfectness and the perfectness are chosen as most characteristic among other ones.

In this paper we accept the following conventions unless otherwise stated:

Matrices and sequences are infinite and complex;

Linear spaces and linear maps between them are over the field $\mathbb{C}$;

All indexes i, k, l, n, m *ranges over natural numbers* $1, 2, \ldots$;

If x *is a sequence, we denote by* x^n *its* n*-th element, i.e.* $x = (x^n)$.

Moreover, if x is convergent, we adopt $x^{\cdot} = \lim x = \lim_n x^n$;

We shall shortly write Σ_i instead of $\Sigma_{i=1}^{\infty}$. For example, the notation $\Sigma_i x_i$ (resp. $\Sigma_i x^i$) means the infinite sum $\Sigma_{i=1}^{\infty} x_i$ (resp. $\Sigma_{i=1}^{\infty} x^i$);

By $\mathbf{1}$ *will be denoted the sequence* $(1, 1, \dots)$;

For any sets A and B the notation $A \subset B$ means that A is a proper subset of B. Moreover, we adopt that $A \subseteq B$ if $A \subset B$ or $A = B$.

We introduce the following notations:

T - *the set of all sequences*;

T_b - *the set of all bounded sequences*;

T_c - *the set of all convergent sequences*;

T_{c0} - *the set of all sequences converging to zero.*

The set T will be regarded as a linear space under the coordinatewise operations. Clearly, the sets T_b, T_c and T_{c0} are linear subspaces of T.

In general, by a *method* we shall mean an object A together with a linear functional (*the limit functional of* A), denoted by $A(\cdot)$, which is determined by this method, i.e. the object A, and defined on a linear subspace $\mathcal{D}(A)$ of T (*the full convergence field of* A). If $x \in \mathcal{D}(A)$, then by $A(x)$ will be denoted the value of $A(\cdot)$ on x. Moreover, if $A(x) = a$, we say that A *limits x (to a)* or that x is A-it limitable (to a). Following [6] we introduce:

$\mathcal{D}_0(A) = \{x \in \mathcal{D}(A) : A(x) = 0\}$;

$\mathcal{D}^b(A) = \mathcal{D}(A) \cap T_b$ - *the bounded convergence field of* A;

$\mathcal{D}_0^b(A) = \{x \in \mathcal{D}^b(A) : A(x) = 0\}$.

For any method A we define the *b-limit functional* of A to be the functional $A^b(\cdot) = A(\cdot)|\mathcal{D}^b(A)$.

The general notion of a method is very useful as the basis for a common language for all special kinds of methods considered in the paper. In particular, this concerns the classical matrix methods, methods defined by iteration products of matrix transformations and SM-methods introduced here.

If A and B are methods such that $\mathcal{D}(A) \subseteq \mathcal{D}(B)$ ($\mathcal{D}^b(A) \subseteq \mathcal{D}^b(B)$), we say that B is a *$\mathcal{D}$-majorant* (*$\mathcal{D}^b$-majorant*) of A or that A is a *$\mathcal{D}$-minorant* (*$\mathcal{D}^b$-minorant*) of B. For any classes $\mathcal{K}$ and $\mathcal{L}$ of methods we introduce the following notations:

$\mathcal{K}^{\leftarrow}(\mathcal{L}) = \{A \in \mathcal{K} : \exists B \in \mathcal{L},\ \mathcal{D}(A) \subseteq \mathcal{D}(B)\}$;

$\mathcal{K}^{\leftarrow b}(\mathcal{L}) = \{A \in \mathcal{K} : \exists B \in \mathcal{L},\ \mathcal{D}^b(A) \subseteq \mathcal{D}^b(B)\}$;

$\mathcal{K}^{\rightarrow}[\mathcal{L}] = \{A \in \mathcal{K} : \forall B \in \mathcal{L},\ \mathcal{D}(B) \subseteq \mathcal{D}(A)\}$;
$\mathcal{K}^{\rightarrow b}[\mathcal{L}] = \{A \in \mathcal{K} : \forall B \in \mathcal{L},\ \mathcal{D}^b(B) \subseteq \mathcal{D}^b(A)\}$.

Methods A and B are said to be *consistent* (*b-consistent*) *over* a set $S \subseteq T$ if $S \subseteq \mathcal{D}(A) \cap \mathcal{D}(B)$ ($S \subseteq \mathcal{D}^b(A) \cap \mathcal{D}^b(B)$) and $A(\cdot)|S = B(\cdot)|S$. Accept that if $S = \mathcal{D}(A) \cap \mathcal{D}(B)$ ($S = \mathcal{D}^b(A) \cap \mathcal{D}^b(B)$), we can omit the indication of S in this definition. If $\mathcal{D}(A) \subseteq \mathcal{D}(B)$ ($\mathcal{D}^b(A) \subseteq \mathcal{D}^b(B)$) and the methods A and B are consistent (b-consistent), we write $A \subseteq B$ ($A \subseteq_b B$). In this case we say that B is a *majorant* (*b-majorant*) of A or that A is a *minorant* (*b-minorant*) of B. Clearly, the relation $\subseteq$ ($\subseteq_b$) between methods is reflexive and transitive, which means that it defines a quasi order, called the *quasi order* (*quasi b-order*) for methods. If $\mathcal{K}$ and $\mathcal{L}$ are classes of methods, we introduce the following notations:

$\mathcal{K}^{\Leftarrow}[\mathcal{L}] = \{A \in \mathcal{K} : \forall B \in \mathcal{L},\ A \subseteq B\}$;
$\mathcal{K}^{\Leftarrow b}[\mathcal{L}] = \{A \in \mathcal{K} : \forall B \in \mathcal{L},\ A \subseteq_b B\}$;
$\mathcal{K}^{\Rightarrow}[\mathcal{L}] = \{A \in \mathcal{K} : \forall B \in \mathcal{L},\ B \subseteq A\}$;
$\mathcal{K}^{\Rightarrow b}[\mathcal{L}] = \{A \in \mathcal{K} : \forall B \in \mathcal{L},\ B \subseteq_b A\}$.

Methods A and B are said to be *equivalent* (*b-equivalent*) if $A \subseteq B$ and $B \subseteq A$ ($A \subseteq_b B$ and $B \subseteq_b A$). For such methods we write $A \simeq B$ ($A \simeq_b B$) and say that B is a *representation* (*b-representation*) of A, and conversely. In this case if B belongs to a class $\mathcal{K}$ of methods, we also say that B is a *$\mathcal{K}$-representation* (*$\mathcal{K}^b$-representation*) of A. Clearly, the relation $\simeq$ ($\simeq_b$) is an equivalence relation between methods and we have $A \simeq B$ ($A \simeq_b B$) if and only if $A(\cdot) = B(\cdot)$ ($A^b(\cdot) = B^b(\cdot)$). It is seen that the quasi order $\subseteq$ (quasi b-order $\subseteq_b$) induces a partial order for the equivalence (b-equivalence) classes of methods, that is, the relation $\subseteq$ ($\subseteq_b$) is antisymmetric up to equivalence (b-equivalence) of methods. For any methods A and B we shall write $A \subset B$ ($A \subset_b B$) if $A \subseteq B$ ($A \subseteq_b B$) and A is not equivalent (b-equivalent) to B.

If $\mathcal{K}$ and $\mathcal{L}$ are classes of methods, we set
$\mathcal{K}^{-}[\mathcal{L}] = \mathcal{K}^{\Leftarrow}[\mathcal{L}] \cap \mathcal{K}^{\Rightarrow}[\mathcal{K}^{\Leftarrow}[\mathcal{L}]]$;
$\mathcal{K}^{-b}[\mathcal{L}] = \mathcal{K}^{\Leftarrow b}[\mathcal{L}] \cap \mathcal{K}^{\Rightarrow b}[\mathcal{K}^{\Leftarrow b}[\mathcal{L}]]$.

It is seen that $\mathcal{K}^{-}[\mathcal{L}]$ ($\mathcal{K}^{-b}[\mathcal{L}]$) denotes the class of all greatest minorants (b-minorants) of $\mathcal{L}$ in $\mathcal{K}$ which may be empty in general. More precisely, $\mathcal{K}^{-}[\mathcal{L}]$ ($\mathcal{K}^{-b}[\mathcal{L}]$) consists of all methods $A \in \mathcal{K}$ satisfying the following conditions:

(i) $A \subseteq B$ ($A \subseteq_b B$) for each $B \in \mathcal{L}$;

(ii) if $A' \in \mathcal{K}$ and $A' \subseteq B$ ($A' \subseteq_b B$) for each $B \in \mathcal{L}$, then $A' \subseteq A$ ($A' \subseteq_b A$).

It is seen that for any $A \in \mathcal{K}^-[\mathcal{L}]$ ($A \in \mathcal{K}^{-b}[\mathcal{L}]$) we have

$$\mathcal{K}^-[\mathcal{L}] = \{A' \in \mathcal{K} : A' \simeq A\}$$

$$(\mathcal{K}^{-b}[\mathcal{L}]) = \{A' \in \mathcal{K} :\ A' \simeq_b A\}),$$

which means that a greatest minorant (b-minorant) of $\mathcal{L}$ in $\mathcal{K}$ is defined up to equivalence (b-equivalence) of methods from $\mathcal{K}$.

A sequence $x \in T$ is called *almost convergent to* x^* if

$$\lim_k \frac{x^n + \cdots + x^{n+k-1}}{k} = x^*$$

uniformly in n, and write $\mathrm{Lim}\, x = \mathrm{Lim}_n\, x^n = x^*$. It is known that every almost convergent sequence is bounded (Lorentz). We introduce the following notations:

T_{ac} - *the set of all almost convergent sequences*;

T_{ac0} - *the set of all sequences almost converging to zero.*

A method A is called *convergence zero-preserving* (*preserving*) if $T_{c0} \subseteq \mathcal{D}(A)$ $(T_c \subseteq \mathcal{D}(A))$. Moreover, such A is called *zero-regular* (*regular*) if $A(x) = 0$ for $x \in T_{c0}$ ($A(x) = \lim x$ for $x \in T_c$). Analogously we define an *almost convergence zero-preserving* (*preserving*) method. By a *strongly zero-regular* (*regular*) method we shall mean an almost convergence zero-preserving (preserving) method A such that $A(x) = 0$ for $x \in T_{ac0}$ ($A(x) = \mathrm{Lim}\, x$ for $x \in T_{ac}$).

A method A is said to be T_{co}-*continuous* (T_c-*continuous*) if it is convergence zero-preserving (preserving) and the limit functional $A(\cdot)$ restricted to T_{c0} (T_c) is continuous in the uniform topology. If A is a T_{c0}-continuous method, we conclude that

$$A(x) = \Sigma_i A_i x^i \quad \text{for} \quad x \in T_{c0},$$

where (A_i) is a complex sequence uniquely defined by A such that $\Sigma_i |A_i| < \infty$. Moreover, if in addition A is convergence preserving or equivalently $\mathbf{1} \in \mathcal{D}(A)$, then it is T_c-continuous and we have

$$A(x) = \Sigma_i A_i (x^i - x^{\cdot}) + x^{\cdot} A(\mathbf{1}) \quad \text{for} \quad x \in T_c.$$

Clearly, if A is zero-regular, then $A_i = 0$ $(i = 1, 2, \dots)$. For any T_c-continuous method A we define the *characteristic of A* to be the complex number $\chi(A) = A(\mathbf{1}) - \Sigma_i A_i$. In this case we have $A(x) = \chi(A)x^{\cdot} + \Sigma_i A_i x^i$ for $x \in T_c$. For example, note that every convergence zero-preserving (preserving) matrix method $\Lambda = (\lambda_i^m)$ is T_{c0}-continuous (T_c-continuous). Moreover, we have $\Lambda_i = \lim_m \lambda_i^m$ $(i = 1, 2, \dots)$, and $\Lambda(\mathbf{1}) = \lim_m \Sigma_i \lambda_i^m$ provided that $\mathbf{1} \in \mathcal{D}(\Lambda)$.

In this paper by $\mathcal{R}$ will be denoted the class of all regular methods. If $\mathcal{K}$ is an arbitrary class of regular methods, then for any $A \in \mathcal{R}^{\leftarrow}(\mathcal{K})$ $\left(A \in \mathcal{R}^{\leftarrow b}(\mathcal{K})\right)$ we define the *perfect part of the full (bounded) convergence field of A relative to $\mathcal{K}$* to be the set

$$\mathcal{D}^p(A;\mathcal{K}) = \{x \in \mathcal{D}(A) : \forall B \in \mathcal{K}^{\rightarrow}[A],\ B(x) = A(x)\}$$
$$\left(\mathcal{D}^{pb}(A;\mathcal{K}) = \{x \in \mathcal{D}^b(A) : \forall B \in \mathcal{K}^{\rightarrow b}[A],\ B(x) = A(x)\}\right)$$

where $\mathcal{K}^{\rightarrow}[A] = \mathcal{K}^{\rightarrow}[\mathcal{L}]$ $(\mathcal{K}^{\rightarrow b}[A] = \mathcal{K}^{\rightarrow b}[\mathcal{L}])$ for $\mathcal{L} = \{A\}$. Note that $T_c \subseteq \mathcal{D}^{pb}(A;\mathcal{K}) \subseteq \mathcal{D}^p(A;\mathcal{K})$ and both $\mathcal{D}^p(A;\mathcal{K})$ and $\mathcal{D}^{pb}(A;\mathcal{K})$ are linear subspaces of T.

A method $A \in \mathcal{R}^{\leftarrow}(\mathcal{K})$ $\left(A \in \mathcal{R}^{\leftarrow b}(\mathcal{K})\right)$ is said to be *perfect* (*b-perfect*) *in $\mathcal{K}$* if $\mathcal{D}^p(A;\mathcal{K}) = \mathcal{D}(A)$ $\left(\mathcal{D}^{pb}(A;\mathcal{K}) = \mathcal{D}^b(A)\right)$. By a *perfectly* (*b-perfectly*) *inconsistent method in $\mathcal{K}$* we shall mean a method $A \in \mathcal{R}^{\leftarrow}(\mathcal{K})$ $\left(A \in \mathcal{R}^{\leftarrow b}(\mathcal{K})\right)$ such that $\mathcal{D}^p(A;\mathcal{K}) = T_c$ $\left(\mathcal{D}^{pb}(A;\mathcal{K}) = T_c\right)$. It follows that for any $A, B \in \mathcal{R}^{\leftarrow}(\mathcal{K})$ $\left(A, B \in \mathcal{R}^{\leftarrow b}(\mathcal{K})\right)$ such that $A \subseteq B$ $(A \subseteq_b B)$ we have $\mathcal{D}^p(A;\mathcal{K}) \subseteq \mathcal{D}^p(B;\mathcal{K})$ $\left(\mathcal{D}^{pb}(A;\mathcal{K}) \subseteq \mathcal{D}^{pb}(B;\mathcal{K})\right)$. In particular, if B is perfectly (b-perfectly) inconsistent, then so is A.

Throughout this paper by $\mathcal{M}$ will be denoted the class of all regular matrix methods. In case $\mathcal{K} = \mathcal{M}$ we shall frequently omit the indication of $\mathcal{M}$ in the above definitions. In particular, we accept $\mathcal{D}^p(A) = \mathcal{D}^p(A;\mathcal{M})$ $\left(\mathcal{D}^{pb}(A) = \mathcal{D}^{pb}(A;\mathcal{M})\right)$ and shortly say that a method is *perfect* (*b-perfect*) or *perfectly* (*b-perfectly*) *inconsistent* if it is such in $\mathcal{M}$.

Let $\mathcal{M}_{\cdot}$ denote the class of all zero-regular convergence preserving matrix methods and let

$$\mathcal{M}_0 = \{A \in \mathcal{M}_{\cdot} : \chi(A) = 0\}.$$

For any method $A \in \mathcal{R}^{\leftarrow}(\mathcal{M})$ $(A \in \mathcal{R}^{\leftarrow b}(\mathcal{M}))$ we introduce the following notations:

$$\mathcal{D}^p_{\cdot}(A) = \{x \in \mathcal{D}(A) : \forall B \in \mathcal{M}^{\rightarrow}_{\cdot}[A],\ B(x) = \chi(B)A(x)\};$$
$$(\mathcal{D}^{pb}_{\cdot}(A) = \{x \in \mathcal{D}(A) : \forall B \in \mathcal{M}^{\rightarrow b}_{\cdot}[A],\ B(x) = \chi(B)A(x)\});$$
$$\mathcal{D}^p_0(A) = \{x \in \mathcal{D}(A) : \forall B \in \mathcal{M}^{\rightarrow}_0[A],\ B(x) = 0\};$$
$$(\mathcal{D}^{pb}_0(A) = \{x \in \mathcal{D}(A) : \forall B \in \mathcal{M}^{\rightarrow b}_0[A],\ B(x) = 0\})$$

where $\mathcal{M}.[A] = \mathcal{M}.[\mathcal{L}]$ $(\mathcal{M}.[A] = \mathcal{M}.[\mathcal{L}])$ and $\mathcal{M}_0[A] = \mathcal{M}_0[\mathcal{L}]$ $(\mathcal{M}_0[A] = \mathcal{M}_0[\mathcal{L}])$ for $\mathcal{L} = \{A\}$.

It is easy to show

0.1. Lemma. *If* $A \in \mathcal{R}^{\leftarrow}(\mathcal{M})$ $(A \in \mathcal{R}^{\leftarrow b}(\mathcal{M}))$, *then*

$$\mathcal{D}^p_{\cdot}(A) = \mathcal{D}^p(A) \subseteq \mathcal{D}^p_0(A) \quad (\mathcal{D}^{pb}_{\cdot}(A) = \mathcal{D}^{pb}(A) \subseteq \mathcal{D}^{pb}_0(A)).$$

Let now $\mathcal{K}$ be an arbitrary class of methods. By a *sequential* $\mathcal{K}$-*method* or shortly an $\mathrm{S}(\mathcal{K})$-*method* we shall mean an infinite sequence $A = (A^n)$ of methods from $\mathcal{K}$ where A^n denotes the n-th element of A. If A is an $\mathrm{S}(\mathcal{K})$-method, we define the *full (bounded) convergence pseudofield of* A to be the set

$$\mathcal{D}^{\sim}(A) = \bigcap_{n=1}^{\infty} \mathcal{D}(A^n) \quad (\mathcal{D}^{\sim b}(A) = \bigcap_{n=1}^{\infty} \mathcal{D}^b(A^n))$$

which is obviously a linear subspace of T (T_b). For such A we denote by $A^{\sim}(\cdot)$ the linear transformation from $\mathcal{D}^{\sim}(A)$ to T given by $A^{\sim}(x) = (A^n(x))$. If S and S' are subsets of T, we say that A transforms S into S' provided that $S \subseteq \mathcal{D}^{\sim}(A)$ and $A^{\sim}(S) \subseteq S'$. The set $\mathcal{D}(A) = A^{\sim -1}(T_c)$ is a linear subspace of T, called the *full convergence field of* A. The *limit functional* $A(\cdot)$ is defined by $A(x) = \lim_n A^n(x)$ for $x \in \mathcal{D}(A)$. Thus, an $\mathrm{S}(\mathcal{K})$-method is a method in the general sense. By an S-*method* we shall mean an $\mathrm{S}(\mathcal{K})$-method where $\mathcal{K}$ is the class of all methods.

For example, if $\Lambda = (\lambda_i^m)$ is a matrix method, then every row $\Lambda^m = (\lambda_1^m, \lambda_2^m, \dots) \in T$ can be regarded as a method with the limit functional $\Lambda^m(\cdot)$ defined by

$$\Lambda^m(x) = \Sigma_i \lambda_i^m x^i \quad \text{for } x \in \mathcal{D}(\Lambda^m),$$

where $\mathcal{D}(\Lambda^m)$ is the set of all $x \in T$ such that the series $\Sigma_i \lambda_i^m x^i$ is convergent. It is seen that Λ can be identified with the S(T)-method (Λ^m). In particular, we have the pseudofield $\mathcal{D}^{\sim}(\Lambda)$ and the linear transformation $\Lambda^{\sim}(\cdot) : \mathcal{D}^{\sim}(\Lambda) \to T$.

Let **M** denote the class of all usual matrix methods. By **SM** we shall denote the class of all S(**M**)-methods which are the main objects under consideration, called SM-*methods* for short. Especially, we are interested in the class $\mathcal{M}^2$ of all regular SM-methods.

Every SM-method A determines a sequence (A^n) of matrix methods such that $A = (A^n)$. This means that A is represented by a unique complex infinite (1,2)-indexed matrix (a_i^{nk}) such that $A^n = (a_i^{nk})$ $(n = 1, 2, \dots)$, which implies

$$A(x) = \lim_n A^n(x) = \lim_n \lim_k \sum_i a_i^{nk} x^i \quad \text{for } x \in \mathcal{D}(A).$$

Let us put

$$A^{nk} = (A^n)^k = (a_i^{nk}) \quad (n = 1, 2, \dots ; \ k = 1, 2, \dots)$$

and note that $A^{nk}(x) = \Sigma_i a_i^{nk} x^i$ for $x \in \mathcal{D}(A^{nk})$. We define the *inner pseudofield of* A to be the set

$$\mathcal{D}^{\approx}(A) = \bigcap_{n=1}^{\infty} \mathcal{D}^{\sim}(A^n) = \bigcap_{n=1}^{\infty} \bigcap_{k=1}^{\infty} \mathcal{D}(A^{nk}).$$

Let $T^2 = \mathbb{C}^{\mathbb{N}\times\mathbb{N}}$ denote the set of all 2-indexed sequences. If $x \in T^2$, then by x^{nk} will be denoted its nk-th element, i.e. $x = (x^{nk})$. For any (1,2)-indexed matrix $A = (a_i^{nk})$, we define the linear transformation $A^{\approx}(\cdot) : \mathcal{D}^{\approx}(A) \to T^2$ by $A^{\approx}(x) = (A^{nk}(x))$. If $S \subseteq T$ and $S' \subseteq T^2$, we say that A transforms S into S' provided that $S \subseteq \mathcal{D}^{\approx}(A)$ and $A^{\approx}(S) \subseteq S'$.

It is easy to prove (compare [5], Theorem 1.3.2)

0.2. Proposition. *Let A be an S-method. If A is convergence zero-preserving (preserving) and each A^n is T_{co}-continuous (T_c-continuous), then so is A. Moreover, A is consistent with some matrix method over T_{co}. If in addition A is convergence preserving and $\lim_n \chi(A^n) = 0$, then A is consistent with some matrix method over T_c.*

From this proposition and since every convergence zero-preserving (preserving) matrix method is T_{co}-continuous (T_c-continuous), we get

0.3. Corollary. *If A is a convergence zero-preserving (preserving) SM-method, then A is T_{c0}-cntinuous (T_c-continuous).*

Let $\Lambda = (\lambda_i^m)$ be a matrix method. It is seen that Λ is equivalent to the SM-method $\Lambda^\uparrow = (a_i^{mk})$ defined by $a_i^{mk} = \lambda_i^m$ ($k = 1,2...$). Moreover, one can see that Λ is equivalent to the SM-method (Λ) defined by $(\Lambda)^n = \Lambda$ ($n = 1,2,...$). On the other hand, for global (b-global) properties of methods one can identify equivalent (b-equivalent) methods. Thus, we can regard that $\mathbf{M} \subseteq \mathbf{SM}$ via the identification $\Lambda \mapsto \Lambda^\uparrow$ ($\Lambda \simeq \Lambda^\uparrow$) or $\Lambda \mapsto (\Lambda)$ ($\Lambda \simeq (\Lambda)$). In this paper we prefer the first identification unless otherwise stated because $\mathcal{D}^{\approx}(\Lambda^\uparrow) = \mathcal{D}^{\sim}(\Lambda^\uparrow) = \mathcal{D}^{\sim}(\Lambda)$ but $\mathcal{D}^{\approx}((\Lambda)) = \mathcal{D}^{\sim}(\Lambda)$ and $\mathcal{D}^{\sim}((\Lambda)) = \mathcal{D}(\Lambda)$, i.e. $\Lambda \mapsto \Lambda^\uparrow$ preserves the pseudofields of methods but $\Lambda \mapsto (\Lambda)$ does not. In particular, we can regard that $\mathcal{M} \subseteq \mathcal{M}^2$. This implies that for any method $A \in \mathcal{R}^{\leftarrow}(\mathcal{M})$ $\left(A \in \mathcal{R}^{\leftarrow b}(\mathcal{M})\right)$ we have

$$\mathcal{D}^p(A;\mathcal{M}^2) \subseteq \mathcal{D}^p(A) \quad \left(\mathcal{D}^{pb}(A;\mathcal{M}^2) \subseteq \mathcal{D}^{pb}(A)\right).$$

Let $\Lambda = (\lambda_i^m)$ and $\Theta = (\theta_m^n)$ be matrices. Recall that the *iteration product* $\Theta[\Lambda]$ *of matrix transformations*, that is, the composition of the matrix transformations Λ and Θ is defined by

$$(*) \qquad z^n = \sum_m \left(\theta_m^n \cdot \Sigma_i \lambda_i^m x^i\right) \quad (n = 1,2,...)$$

for $x \in \mathcal{D}^{\sim}(\Theta[\Lambda])$, the *convergence pseudofield of* $\Theta[\Lambda]$ consisting of all $x \in T$ such that the series are convergent. This means that $\Theta[\Lambda]^{\sim} : \mathcal{D}^{\sim}(\Theta[\Lambda]) \to T$ is a linear map given by (*), where $z = \Theta[\Lambda](x)$. The *convergence field of* $\Theta[\Lambda]$ is defined to be the set $\mathcal{D}(\Theta[\Lambda]) = \Theta[\Lambda]^{\sim -1}(T_c)$. Finally, the *limit functional* $\Theta[\Lambda](\cdot)$ is given by $\Theta[\Lambda](x) = \lim \Theta[\Lambda]^{\sim}(x)$ for $x \in \mathcal{D}(\Theta[\Lambda])$. The method defined above will be called the IPM-*method* $\Theta[\Lambda]$.

Let $\Lambda = (\lambda_i^m)$ and $\Theta = (\theta_m^n)$ be matrices. Following Agnew [1] the *composition product of* Λ *and* Θ is defined to be the matrix $\Theta\Lambda = (c_i^n)$ where

$$c_i^n = \Sigma_m \theta_m^n \lambda_i^m \quad (n = 1,2,...;\ i = 1,2,...)$$

provided that all series are convergent. It is known that if the IPM-method $\Theta[\Lambda]$ is convergence zero-preserving (see [8], Theorem I.1), then the composition product $\Theta\Lambda$ exists. It turns out that in general the methods $\Theta\Lambda$ and $\Theta[\Lambda]$ need not be equivalent (b-equivalent), even if both Θ and Λ are regular.

1. Basic properties

1.1. Equivalence between IPM-methods and SM-methods.

Let $\Lambda = (\lambda_i^m)$ and $\Theta = (\theta_m^n)$ be matrices. Define the (1,2)-indexed matrix $\Theta\#\Lambda = (c_i^{nk})$ as follows:

$$c_i^{nk} = \begin{cases} \lambda_i^n & \text{for } n = 1,2,\dots;\ k = 1, \\ \sum_{m=1}^{k-1} \theta_m^n \lambda_i^m & \text{for } n = 1,2,\dots;\ k = 2,3,\dots \end{cases}.$$

This matrix determines a unique SM-method, denoted also by $\Theta\#\Lambda$, such that $\Theta\#\Lambda = (C^n)$ where $C^n = (c_i^{nk})$ for each n. One can show

1.1.1. Lemma. *For any matrices Λ and Θ the IPM-method $\Theta[\Lambda]$ is equivalent to the SM-method $\Theta\#\Lambda$.*

Let now $C = (c_i^{nk})$ be a $(1,2)$-indexed matrix. Let us take a one-to-one map τ from $\mathbb{N}^2$ onto $\mathbb{N}$ such that $\tau(n,k) < \tau(n,k+1)$ for all $n,k \in \mathbb{N}$. For example, such a function can be given by $\tau(n,k) = 2^n k - 2^{n-1}$. Define matrices $\Lambda = (\lambda_i^m)$ and $\Theta = (\theta_m^n)$ as follows:

(1.1α)

$$\lambda_i^m = \begin{cases} c_i^{n1} & \text{for } m = \tau(n,1)\ (n = 1,2,\dots) \\ c_i^{nk} - c_i^{n,k-1} & \text{for } m = \tau(n,k)\ (n = 1,2,\dots; \\ & \qquad k = 2,3,\dots) \end{cases}$$

(1.1β)

$$\theta_m^n = \begin{cases} 0 & \text{if } m \neq \tau(n,k) \text{ for each } k = 1,2,\dots \\ 1 & \text{if } m = \tau(n,k) \text{ for some } k = 1,2,\dots \end{cases}$$

By an easy verification we get

1.1.2. Lemma. *If $C = (c_i^{nk})$ is a $(1,2)$-indexed matrix and if matrices $\Lambda = (\lambda_i^m)$ and $\Theta = (\theta_m^n)$ are defined by formulas (1.1α) and (1.1β), then the method C is equivalent to the IPM-method $\Theta[\Lambda]$.*

Clearly, Lemmas 1.1.1 and 1.1.2 imply

1.1.3. Proposition. *Every IPM-method is equivalent to an SM-method, and conversely.*

Let $\Lambda = (\lambda_i^m)$ and $\Theta = (\theta_m^n)$ be matrices. As we know from Chapter 0 the composition product $\Theta\Lambda$ exists provided that the IPM-method $\Theta[\Lambda]$ is convergence zero-preserving. Additionally, one can see that if Λ (Θ) has finite columns (rows), then $\Theta\Lambda$ exists too. It is easy to prove the following propositions.

1.1.4. Proposition. *If $\Lambda = (\lambda_i^m)$ and $\Theta = (\theta_i^n)$ are matrices such that the composition product $\Theta\Lambda$ exists and if the following condition holds:*

$$\lim_k \sum_i \Big| \sum_{m=k}^{\infty} \theta_m^n \, \lambda_i^m \Big| = 0 \quad (n = 1, 2, \dots),$$

then the methods $\Theta\Lambda$ and $\Theta[\Lambda]$ are b-equivalent.

1.1.5. Proposition. *If $\Lambda = (\lambda_i^m)$ and $\Theta = (\theta_i^n)$ are matrices and if Θ has finite rows, then the methods $\Theta\Lambda$ and $\Theta[\Lambda]$ are equivalent.*

1.2. Regularity and limitation of bounded sequences.

For any matrix $\Lambda = (\lambda_i^m)$ we shall adopt the following conditions:

(1.2Λ) there exist limits $\lim_m \lambda_i^m = \lambda_i (i = 1, 2, \dots)$;

(1.2Θ) $\sup_m \sum_i |\lambda_i^m| < \infty$;

(1.2Γ) there exists a limit $\lim_m \sum_i \lambda_i^m = \lambda$.

The following lemma is well known.

1.2.1. Lemma. *A matrix method $\Lambda = (\lambda_i^m)$ is convergence zero-preserving (preserving) if and only if it satisfies conditions (1.2Λ)-(1.2Θ) ((1.2Λ)-(1.2Γ)). Moreover, the following statements hold:*

(1) $\lim_m \sum_i \lambda_i^m x^i = \sum_i \lambda_i x^i$ *for each* $x \in T_{co}$ *provided that* Λ *is convergence zero-preserving;*

(2) $\lim_m \sum_i \lambda_i^m x^i = (\lambda - \sum_i \lambda_i)\, x^{\cdot} + \sum_i \lambda_i x^i$ *for each* $x \in T_c$ *provided that* Λ *is convergence preserving.*

Let now $A = (a_i^{nk})$ be a $(1,2)$-indexed matrix. Adopt the following conditions:

(1.2a) there exist limits $\lim_n \lim_k a_i^{nk} = a_i$ $(i = 1, 2, \dots)$;

(1.2b) $\sup_k \sum_i |a_i^{nk}| < \infty$ $(n = 1, 2, \dots)$;

(1.2c) there exist limits $\lim_k a_i^{nk} = a_i^n$ $(n = 1, 2, \dots;\ i = 1, 2, \dots)$ and $\sup_n \sum_i |a_i^n| < \infty$;

(1.2d) there exist a limit $\lim_n \lim_k \sum_i a_i^{nk} = a$.

We introduce the following notations:

(1.2e) $A^0 = (a_i^{nk} - a_i)$ and $A^{\cdot 0} = (a_i^n - a_i)$ provided that condition (1.2a) holds;

(1.2f) $A^{\cdot} = (a_i^n)$ and $A^{n0} = (a_i^{nk} - a_i^n)$ $(n = 1, 2, \dots$ provided that condition (1.2c) holds.

Note that if A satisfies conditions (1.2b) and (1.2c) ((1.2a) and (1.2c)), then each matrix method A^n $(A^{\cdot})$ is convergence zero-preserving. Clearly, in this case the methods A^{n0} and $A^{\cdot 0}$ are zero-regular. Moreover, if in addition A satisfies condition (1.2d), then each A^n is convergence preserving.

Applying Lemma 1.2.1 one can prove

1.2.2. Theorem. *An SM-method $A = (a_i^{nk})$ is convergence zero-preserving (preserving) if and only if it satisfies conditions (1.2a)-(1.2c) ((1.2a)-(1.2d)). Moreover, the following statements hold:*

(1) $\lim_n \lim_k \sum_i a_i^{nk} x^i = \sum_i a_i x^i$ *for each* $x \in T_{c0}$ *provided that* A *is convergence zero-preserving;*

(2) $\lim_n \lim_k \sum_i a_i^{nk} x^i = (a - \sum_i a_i)\, x^{\cdot} + \sum_i a_i x^i$ *for each* $x \in T_c$ *provided that* A *is convergence preserving.*

This theorem implies the following observations. If A is an SM-method such that $A^{\cdot}$ exists and if all A^n and $A^{\cdot}$ are convergence zero-preserving, then so is A. If in addition all A^n and $A^{\cdot}$ are convergence preserving, then A is convergence preserving if and only if $(\chi(A^n)) \in T_c$. In particular, if all A^n are convergence zero-preserving and if $A^{\cdot}$ is zero-regular, then so is A. Moreover, if all A^n are convergence preserving and if $\lim_n \chi(A^n) = 0$ and $A^{\cdot}$ is regular, then A is regular too.

Clearly, if A is convergence zero-preserving, then notations (1.2e) and (1,2f) have meaning.

Additionally, we adopt the following conditions:

(1.2′) the limits $\lim_n \lim_k a_i^{nk} = 0$ exist;
(1.2d′) the limit $\lim_n \lim_k \sum_i a_i^{nk} = 1$ exists.

Theorem 1.2.2 immediately implies

1.2.3. Corollary. *An SM-method $A = (a_i^{nk})$ is zero-regular (regular) if and only if it satisfies conditions (1.2a′),(1.2b) and (1.2c) ((1.2a′),(1.2d′) and (1.2b),(1.2c)).*

Note that this corollary involves that A is zero-regular if and only if each A^n is convergence zero-preserving and $A^{\cdot}$ is zero-regular. Moreover, A is regular if and only if A satisfies (1,2d′), each A^n is convergence preserving and $A^{\cdot}$ is zero-regular.

By Corollary 0.2, every convergence preserving SM-method A is T_c-continuous, and so, the characteristic $\chi(A)$ is defined. Moreover, in this case each A^n (A^{no}) is T_c-continuous too. Clearly, we have

$$A^n(x) = \chi(A^n)\, x^{\cdot} + \sum_i a_i^n\, x^i \quad \text{for} \quad x \in T_c \ \ (n = 1, 2, \dots).$$

From this and Theorem 1.2.2.(2) we get

1.2.4. Corollary. *Let $A = (a_i^{nk})$ be a convergence preserving SM-method. Then*

$$\limsup_n \chi(A^n) = a - \liminf_n \sum_i a_i^n \quad \text{and}$$

$$\liminf_n \chi(A^n) = a - \limsup_n \sum_i a_i^n$$

which implies that $(\chi(A^n))$ *is convergent if and only if* $A^{\cdot}$ *is convergence preserving. In particular, we have*

$$\limsup_n |\chi(A^n)| \leq |a| + \sup_n \sum_i |a_i^n|,$$

and so, $(\chi(A^n)) \in T_b$. *Moreover, if in addition* $A^{\cdot}$ *is convergence preserving, then*

$$\lim_n \chi(A^n) = \chi(a) - \chi(A^{\cdot}).$$

Let $\Lambda = (\lambda_i^m)$ and $\Theta = (\theta_m^n)$ be matrices. Adopt the following conditions:

(1.2λ) there exist limits $\lim_n \sum_m \theta_m^n \lambda_i^m = \gamma_i \ (i = 1, 2, \dots)$;
(1.2θ) $\sum_i \lambda_i^m < \infty \ (m = 1, 2, \dots)$;
(1.2γ) $\sup_k \sum_i \left|\sum_{m=1}^k \theta_m^n \lambda_i^m\right| < \infty \ (n = 1, 2, \dots)$;
(1.2δ) $\sup_n \sum_i \left|\sum_m \theta_m^n \lambda_i^m\right| < \infty$;
(1.2ε) there exists a limit $\lim_n \sum_m \theta_m^n (\sum_i \lambda_i^m) = \gamma$.

By Lemma 1.1.1 and by applying Theorem 1.2.2 to the SM-method $\Theta \# \Lambda$, we get

1.2.5. Corollary. (see [8], Theorems III.1 and III.2) *Let* $\Lambda = (\lambda_i^m)$ *and* $\Theta = (\theta_m^n)$ *be matrices. Then the IPM-method* $\Theta[\Lambda]$ *is convergence zero-preserving (preserving) if and only if the matrices* Λ *and* Θ *satisfy conditions (1.2λ)-(1.2δ) ((1.2λ)-(1.2ε)). Moreover, the following statements hold:*

(1) $\lim_n \sum_m \theta_m^n (\sum_i \lambda_i^m x^i) = \sum_i \gamma_i x^i$ *for each* $x \in T_{c0}$ *provided that* $\Theta[\Lambda]$ *is convergence zero-preserving;*
(2) $\lim_n \sum_m \theta_m^n (\sum_i \lambda_i^m x^i) = (\gamma - \sum_i \gamma_i) x^{\cdot} + \sum_i \gamma_i x^i$ *for each* $x \in T_c$ *provided that* $\Theta[\Lambda]$ *is convergence preserving.*

Well known is the following

1.2.6. Lemma. (Schur) *A matrix method* $\Lambda = (\lambda_i^m)$ *limits all bounded sequences if and only if* Λ *is convergence zero-preserving and the following condition holds:*

$$\lim_m \sum_i |\lambda_i^m - \lambda_i| = 0.$$

Moreover, if $T_b \subseteq \mathcal{D}(\Lambda)$, then

$$\lim_m \sum_i \lambda_i^m x^i = \sum_i \lambda_i x^i \quad \text{for each} \quad x \in T_b.$$

Applying this lemma we obtain

1.2.7. Theorem. *An SM-method $A = (a_i^{nk})$ limits all bounded sequences if and only if so are all A^n and $A^{\cdot}$, that is, the following conditions hold:*

(1) $\lim_k \sum_i |a_i^{nk} - a_i^n| = 0 \quad (n = 1, 2, \dots)$;
(2) $\lim_n \sum_i |a_i^n - a_i| = 0$.

Moreover, if $T_b \subseteq \mathcal{D}(A)$, then

(iii) $\lim_n \lim_k \sum_i a_i^{nk} x^i = \sum_i a_i x^i$ *for each* $x \in T_b$.

This theorem implies that if a zero-regular SM-method A limits all bounded sequences, then each such sequence is A-limitable to 0. In particular, does not exist a regular SM-method which limits all bounded sequences.

Similarly as for Corollary 1.2.5, from Lemma 1.1.1 and Theorem 1.2.7 it follows

1.2.8. Corollary. (see [8], Theorem III.3) *Let $\Lambda = (\lambda_i^m)$ and $\Theta = (\theta_m^n)$ be matrices. Then the IPM-method $\Theta[\Lambda]$ limits all bounded sequences if and only if conditions (1.2λ)-(1.2δ) and the following ones hold:*

(1) $\lim_k \sum_i |\sum_{m=k}^{\infty} \theta_m^n \lambda_i^m| = 0 \quad (n = 1, 2, \dots)$;
(2) $\lim_n \sum_i |\sum_m \theta_m^n \lambda_i^m - \gamma_i| = 0$.

Moreover, if $T_b \subseteq \mathcal{D}(\Theta[\Lambda])$, then

(iii) $\lim_n \sum_m \theta_m^n \left(\sum_i \lambda_i^m x^i\right) = \sum_i \gamma_i x^i$ *for each* $x \in T_b$.

1.3. Strong regularity.

Recall that a matrix $\Lambda = (\lambda_i^m)$ is said to be *translative* in case $\lim_m \sum_i (\lambda_i^m - \lambda_{i+1}^m) x^i = 0$ for all $x \in T_b$, which by Lemma 1.2.6 is equivalent to the fact that

$$\lim_m \sum_i \left|\lambda_i^m - \lambda_{i+1}^m\right| = 0.$$

We need the following well-known

1.3.1. Lemma. *Let $\Lambda = (\lambda_i^m)$ be a convergence zero-preserving (preserving) matrix method. Then Λ is almost convergence zero-preserving (preserving) if and only if the matrix $\Lambda^0 = (\lambda_i^m - \lambda_i)$ is translative. Moreover, the following statements hold:*

(1) *$\lim_m \sum_i \lambda_i^m x^i = \sum_i \lambda_i x^i$ for each $x \in T_{ac0}$ provided that Λ is almost convergence zero-preserving;*

(2) *$\lim_m \sum_i \lambda_i^m x^i = (\lambda - \sum_i \lambda_i)\, x^* + \sum_i \lambda_i x^i$ for each $x \in T_{ac}$ provided that Λ is almost convergence preserving.*

Applying this lemma we get

1.3.2. Theorem. *Let $A = (a_i^{nk})$ be a convergence zero-preserving (preserving) SM-method. Then A is almost convergence zero-preserving (preserving) if and only if all A^{n0} and $A^{\cdot 0}$ are translative. Moreover, the following statements hold:*

(1) *$\lim_n \lim_k \sum_i a_i^{nk} x^i = \sum_i a_i x^i$ for each $x \in T_{ac0}$ provided that A is almost convergence zero-preserving;*

(2) *$\lim_n \lim_k \sum_i a_i^{nk} x^i = (a - \sum_i a_i)\, x^* + \sum_i a_i x^i$ for each $x \in T_{ac}$ provided that A is almost convergence preserving.*

This theorem immediately implies

1.3.3. Corollary. *An SM-method is strongly zero-regular (regular) if and only if it is zero-regular (regular) and almost convergence zero-preserving (preserving).*

An SM-method $A = (a_i^{nk})$ is called *translative* if

$$\lim_n \lim_k \sum_i \left(a_i^{nk} - a_{i+1}^{nk}\right) x^i = 0 \quad \text{for} \quad x \in T_b.$$

By Lemma 1.2.6, we conclude that a convergence zero-preserving SM-method A is translative if and only if all A^{n0} and $A^{\cdot}$ are translative. Hence and from Theorem 1.3.2 we get the following corollaries.

1.3.4. Corollary. *Let A be a convergence zero-preserving (preserving) SM-method. Then A is almost convergence zero-preserving (preserving) if and only if A^0 is translative.*

1.3.5. Corollary. *Let A be a convergence zero-preserving (preserving) SM-method. Then A is strongly zero-regular (regular) if and only if it is translative (translative and regular). In particular, a zero-regular (regular) SM-method is strongly zero-regular (regular) if and only if it is translative.*

Let $\Lambda = (\lambda_i^m)$ and $\Theta = (\theta_m^n)$ be matrices. The IPM-method $\Theta[\Lambda]$ is called *translative* if so is the method $\Theta\#\Lambda$. It follows that $\Theta[\Lambda]$ is translative if and only if the following conditions hold:

(1.3λ)

$$\sum_i |\lambda_i^m - \lambda_{i+1}^m| < \infty \quad (m = 1, 2, \dots),$$

(1.3θ)

$$\lim_k \sum_i \left| \sum_{m=k}^{\infty} \theta_m^n \left(\lambda_i^m - \lambda_{i+1}^m\right) \right| = 0 \quad (n = 1, 2, \dots),$$

(1.3γ)

$$\lim_n \sum_i \left| \sum_m \theta_m^n \left(\lambda_i^m - \lambda_{i+1}^m\right) \right| = 0.$$

Clearly, if $\Theta[\Lambda]$ is convergence zero-preserving, then it is translative if and only if conditions (1.3θ) and (1.3γ) hold. From this and the definition of IPM-method we get

1.3.6. Proposition. *Let Λ and Θ be matrices. If Λ is translative and if Θ is zero-regular, then the IPM-method $\Theta[\Lambda]$ is translative. In particular, if Λ is almost convergence zero-preserving (preserving) and if Θ is zero-regular and convergence preserving, then $\Theta[\Lambda]$ is almost convergence zero-preserving (preserving). Moreover, if Λ is strongly zero-regular (regular) and if Θ is zero-regular (regular), then $\Theta[\Lambda]$ is strongly zero-regular (regular).*

If condition (1.2λ) holds, we adopt the following one:

$$(1.3\gamma^+) \qquad \lim_n \sum_i \left| \sum_m \theta_m^n \left(\lambda_i^m - \lambda_{i+1}^m\right) - \gamma_i + \gamma_{i+1} \right| = 0.$$

It is easily seen that if $\Theta[\Lambda]$ is convergence zero-preserving, then the SM-method $(\Theta\#\Lambda)^0$ is translative if and only if conditions (1.3θ) and $(1.3\gamma^+)$ hold. From this, Theorem 1.3.2 and Corollary 1.3.4 we obtain

1.3.7. Corollary. *Let $\Lambda = (\lambda_i^m)$ and $\Theta = (\theta_m^n)$ be matrices. Suppose that the IPM-method $\Theta[\Lambda]$ is convergence zero-preserving (preserving). Then $\Theta[\Lambda]$ is almost convergence zero-preserving (preserving) if and only if the SM-method $(\Theta\#\Lambda)^0$ is translative, that is, if conditions (1.3θ) and $(1.3\gamma^+)$ hold. Moreover, the following statements hold:*

(1) $\lim_n \sum_m \theta_m^n (\sum_i \lambda_i^m x_i) = \sum_i \gamma_i x^i$ *for each* $x \in T_{ac0}$ *provided that $\Theta[\Lambda]$ is almost convergence zero-preserving;*
(2) $\lim_n \sum_m \theta_m^n (\sum_i \lambda_i^m x_i) = (\gamma - \sum_i \gamma_i)\, x^* + \sum_i \gamma_i x^i$ *for each* $x \in T_{ac}$ *provided that $\Theta[\Lambda]$ is almost convergence preserving.*

Włodarski ([8], p.351) gave an example of matrices Λ and Θ such that the IPM-method $\Theta[\Lambda]$ is regular, $\mathcal{D}(\Theta[\Lambda]) = \mathcal{D}^b(\Theta[\Lambda])$ and $\Theta[\Lambda]$ is not equivalent to the standard convergence. On the other hand, the Mazur-Orlicz Theorem (see [4], Theorem 7) says that if a regular matrix method limits some bounded divergent sequence, then it limits some unbounded sequence. It turns out that this theorem has an analogue for almost convergence zero-preserving SM-methods.

1.3.8. Theorem. *Every almost convergence zero-preserving SM-method limits some unbounded sequence.*

The last theorem implies

1.3.9. Corollary. *There is no SM-method equivalent to the almost convergence.*

The well-known Lorentz Theorem says that there is no matrix method b-equivalent to the almost convergence. One may ask whether there exists an SM-method b-equivalent to the almost convergence. A negative answer to this question will be given in Section 2.5.

1.4. Structure of SM-methods.

By an SM$^{\cdot}$-*method* we shall mean an SM-method A such that each A^n is a convergence zero-preserving matrix method. For every SM$^{\cdot}$-method A the matrix method $A^{\cdot}$ is defined, which means that there exist limits: $a_i^n = \lim_k a_i^{nk}$ $(n = 1, 2, \dots;\ k = 1, 2, \dots)$ and $\sum_i |a_i^n| < \infty$ $(n = 1, 2, \dots)$. Note that if A is an SM$^{\cdot}$-method, then A is convergence zero-preserving (zero-regular) if and only if so is $A^{\cdot}$. Clearly, in this case A is consistent with $A^{\cdot}$ over T_{c0}.

Let $S \subseteq T$. An SM-method A is said to be *decomposable over* S if it is an SM$^{\cdot}$-method and if $S \subseteq \mathcal{D}^{\sim}(A) \cap \mathcal{D}^{\sim}(A^{\cdot})$, i.e. the following decompositions hold:

$$A^n(x) = A^{n0}(x) + A^{n\cdot}(x) \quad \text{for} \quad x \in S \quad (n = 1, 2, \dots).$$

It is seen that every SM$^{\cdot}$-method A is decomposable over $\mathcal{D}^b(A)$. We say that A is *decomposable* if it is decomposable over $\mathcal{D}(A)$.

An SM-method A is called *reducible over* S if there is an SM-representation of A which is decomposable over S. We say that A is *reducible* if it is reducible over $\mathcal{D}(A)$; otherwise A is called *irreducible*. It is easy to give examples of reducible SM-methods but the existence of irreducible SM-methods will be explained in Section 3.5.

A convergence zero-preserving matrix method $\Lambda = (\lambda_i^m)$ is said to be *decomposable over* S if $S \subseteq \mathcal{D}(\Lambda) \cap \mathcal{D}(\Lambda^{\cdot})$, i.e. the following decomposition holds:

$$\Lambda(x) = \Lambda^0(x) + \Lambda^{\cdot}(x) \quad \text{for} \quad x \in S,$$

where $\Lambda^0 = (\lambda_i^m - \lambda_i)$ and $\Lambda^{\cdot} = (\lambda_i)$. Moreover, Λ is said to be *decomposable* if it is decomposable over $\mathcal{D}(\Lambda)$. Clearly, Λ is decomposable if and only if $\mathcal{D}(\Lambda) \subseteq \mathcal{D}(\Lambda^{\cdot})$. It is seen that Λ is decomposable (over S) if and only if so is the SM-method (Λ). Notice however that every SM-method equivalent to a matrix method $\Lambda = (\lambda_i^m)$ satisfying $\Sigma_i |\lambda_i^m| < \infty$ is reducible. Indeed, observe that such a method is equivalent to the SM$^{\cdot}$-method $\Lambda^{\uparrow}$ which is obviously decomposable over $\mathcal{D}(\Lambda)$.

An SM-method is said to be *absolutely reducible* if every SM-representation of it is decomposable. By an easy verification we get

1.4.1. Proposition. *An SM-method A is absolutely reducible if and only if each matrix method Θ such that $\mathcal{D}(A) \subseteq \mathcal{D}(\Theta)$ is decomposable over $\mathcal{D}(A)$.*

Observe that every convergence zero-preserving SM-method A such that $\mathcal{D}(A) = \mathcal{D}^b(A)$ is absolutely reducible. This property is also a consequence of the more general

1.4.2. Proposition. *If A is a convergence zero-preserving SM-method such that $\mathcal{D}(A)$ is closed under multiplication by sequences converging to zero, then A is absolutely reducible.*

An SM-method A is said to be *quasi decomposable* if there exists a subsequence $(A^{n(l)})$ of (A^n) defining the SM-method which is equivalent to A and decomposable. If A is not quasi decomposable, we call it *indecomposable*. Clearly, every SM-representation of an irreducible SM-method is indecomposable. An indecomposable SM-method A is said to be *strongly indecomposable* if there does not exist a subsequence $(A^{n(l)})$ of (A^n) such that the SM-method $(A^{n(l)})$ is decomposable. It is easy to prove

1.4.3. Proposition. *A reducible SM-method A has a strongly indecomposable SM-representation if and only if it is not absolutely reducible.*

An SM-method A is said to be *free* if it is an SM$^{\cdot}$-method such that $A^{\cdot}$ is the zero matrix, that is, each A^n is zero-regular. Obviously, every free SM-method is zero-regular. By a *semi-simple* SM-method we shall mean a free SM-method A such that each A^n is regular. We say that A is *simple* if it is semi-simple and $A^1 \supseteq A^2 \supseteq \ldots \supseteq A^n \supseteq \ldots$. It turns out that one can construct semi-simple SM-methods which are not b-equivalent to any simple SM-methods, and so, they are not b-equivalent to any matrix methods.

By an easy verification we get

1.4.4. Proposition. *Every free regular SM-method has a semi-simple SM-representation.*

One can prove

1.4.5. Theorem. *If A is a free SM-method and Θ is a zero-regular matrix method such that $D(A) \subseteq \mathcal{D}(\Theta)$, then there exists a decomposable SM-method C such that $C \simeq A$ and $C^{\cdot} = \Theta$.*

This theorem immediately implies

1.4.6. Corollary. *If Λ and Θ are zero-regular matrix methods such that $\mathcal{D}(\Lambda) \subseteq \mathcal{D}(\Theta)$, then there exists a decomposable SM-method C such that $C \simeq \Lambda$ and $C^{\cdot} = \Theta$.*

Applying Theorem 1.4.5 it is easy to prove

1.4.7. Corollary. *If A is a free SM-method and Δ is a zero-regular matrix method such that $\mathcal{D}(A) \subseteq \mathcal{D}(\Delta)$, then there exist matrices Λ and Θ such that $\Theta[\Lambda] \simeq A$ and $\Theta\Lambda = \Delta$.*

1.5. Bounded convergence fields.

Recall that T_b denotes the complex Banach space of all bounded sequences under the coordinatewise operations and the norm defined by $\|x\| = \sup_n |x^n|$. We say that a matrix (matrix method) Λ is T_b*-continuous* if $T_b \subseteq \mathcal{D}^{\sim}(\Lambda)$, $\Lambda^{\sim}(T_b) \subseteq T_b$ and the transformation $\Lambda^{\sim}(\cdot)$ restricted to T_b is continuous under the uniform topology on T_b. Clearly, if Λ is T_b-continuous, then $\mathcal{D}^b(\Lambda)$ and $\mathcal{D}^b_0(\Lambda)$ are Banach subspaces of T_b.

Adopt the following notations:

$$T^2_c = \{x \in T^2 \ : \text{the limit } \lim_n \lim_k x^{nk} \text{ exists}\};$$

$$T^2_{c0} = \{x \in T^2 \ : \text{the limit } \lim_n \lim_k x^{nk} = 0 \text{ exists}\};$$

$$T^2_b = \{x \in T^2 \ : \sup_n \sup_k \left|x^{nk}\right| < \infty\};$$

$$T^2_{bc} = T^2_b \cap T^2_c;$$

$$T^2_{bc0} = T^2_b \cap T^2_{c0}.$$

We shall regard T_b^2 as a complex Banach space under the coordinatewise operations and the norm defined by

$$\|x\| = \sup_n \sup_k |x^{nk}|,$$

which determines the uniform topology on T_b^2. Clearly, T_{bc}^2 and T_{bc0}^2 are non-separable Banach subspaces of T_b^2. A (1,2)-indexed matrix (SM-method) A is said to be *T_b-continuous* if $T_b \subseteq \mathcal{D}^{\sim}(A)$, $A^{\sim}(T_b) \subseteq T_b^2$ and $A^{\sim}(\cdot)$ restricted to T_b is continuous under the uniform topologies on T_b and T_b^2. It turns out that if an SM-method A is T_b-continuous, then $\mathcal{D}^b(A) = A^{\sim -1}(T_{bc}^2)$ and $\mathcal{D}_0^b(A) = A^{\sim -1}(T_{bc0}^2)$, which implies that $\mathcal{D}^b(A)$ and $\mathcal{D}_0^b(A)$ are Banach subspaces of T_b (see Theorem 1.5.2 and Corollary 1.5.3).

Well-known is the following

1.5.1. Lemma. (see [5], Theorem 1.3.2) *A matrix $\Lambda = (\lambda_i^m)$ transforms T_b into itself if and only if the following condition holds:*

$$(1.5\lambda) \qquad \sup_m \sum_i |\lambda_i^m| < \infty.$$

Note that if a matrix $\Lambda = (\lambda_i^m)$ satisfies condition (1.5λ), then it is T_b-continuous. In particular, if Λ is convergence zero-preserving, then condition (1.5λ is fulfilled, and so, $\mathcal{D}^b(\Lambda)$ and $\mathcal{D}_o^b(\Lambda)$ are Banach subspaces of T_b. Give attention that an analogous property for SM-methods does not satisfy (see Theorem 1.5.6).

We introduce the following conditions:

$$(1.5a) \qquad \sup_n \sup_k \sum_i |a_i^{nk}| < \infty;$$

$$(1.5b) \qquad \sup_k \sum_i |a_i^{nk}| < \infty \quad (n = 1, 2, \ldots);$$

$$(1.5c) \qquad \sup_n \limsup_k \sum_i |a_i^{nk}| < \infty.$$

One can prove

1.5.2. Theorem. *Let $A = (a_i^{nk})$ be a (1,2)-indexed matrix. Then the following statements are equivalent:*

(a) *A transforms T_b into T_b^2;*
(b) *A satisfies condition (1.5a);*
(c) *A is T_b-continuous.*

This theorem immediately implies

1.5.3. Corollary. *If a matrix $A = (a_i^{nk})$ satisfies condition (1.5a), then $\mathcal{D}^b(A)$ and $\mathcal{D}_0^b(A)$ are Banach subspaces of T_b.*

It is easy to prove

1.5.4. Lemma. *If $A = (a_i^{nk})$ is an SM-method satisfying conditions (1.5b) and (1.5c), then it is b-equivalent to some SM-method satisfying condition (1.5a).*

From this lemma and Corollary 1.5.3 we get

1.5.5. Corollary. *If $A = (a_i^{nk})$ is an SM-method satisfying conditions (1.5b) and (1.5c), then $\mathcal{D}^b(A)$ and $\mathcal{D}_0^b(A)$ are Banach subspaces of T_b.*

Note that a (1,2)-indexed matrix A satisfying conditions (1.5b) and (1.5c) transforms T_b into T^2 but it need not transform some sequences from T_b into T_b^2, even if A is a regular SM-method (compare Corollary 1.2.3).

The following theorem means that in general the bounded convergence field of a regular SM-method cannot be investigated by using Banach space theory.

1.5.6. Theorem. *There are regular SM-methods such that the sets $\mathcal{D}^b(A)$ and $\mathcal{D}_0^b(A)$ are non-separable and non-complete subspaces of T_b, and so, they are not Banach spaces.*

Since bounded convergence field of every regular matrix method is a Banach space, it follows that a regular SM-method satisfying Theorem 1.5.6 is not b-equivalent to any regular matrix method.

Let $\Lambda = (\lambda_i^m)$ and $\Theta = (\theta_m^n)$ be matrices such that

$$\sup_m \sum |\lambda_i^m| < \infty \quad \text{and} \quad \sup_n \sum_i |\theta_m^n| < \infty.$$

It follows from Lemma 1.5.1 that Λ and Θ are T_b-continuous. This implies that the iteration product $\Theta[\Lambda]$ is T_b-continuous too, and so, $\mathcal{D}^b(\Theta[\Lambda])$ and $\mathcal{D}_0^b(\Theta[\Lambda])$ are Banach subspaces of T_b. The same result we get by applying Theorem 1.5.2 to the SM-method $\Theta\#\Lambda$ which is b-equivalent to the IPM-method $\Theta[\Lambda]$. On the other hand, applying Corollary 1.5.5 to the SM-method $\Theta\#\Lambda$ one has

1.5.7. Corollary. *If $\Lambda = (\lambda_i^m)$ and $\Theta = (\theta_m^n)$ are matrices satisfying conditions (1.2θ), (1.2γ) and the following one:*

$$\sup_n \limsup_k \sum_i \left| \sum_{m=1}^{k} \theta_m^n \, \lambda_i^m \right| < \infty \quad (n = 1, 2, \dots),$$

then $\mathcal{D}^b(\Theta[\Lambda])$ and $\mathcal{D}_0^b(\Theta[\Lambda])$ are Banach subspaces of T_b.

2. Perfectness for bounded convergence fields

2.1. Equivalence of b-perfectness in the classes $\mathcal{M}^2$ and $\mathcal{M}$.

As we know from Chapter 0, for any method $A \in \mathcal{R}^{\leftarrow b}(\mathcal{M})$ we have $\mathcal{D}^{pb}(A; \mathcal{M}^2) \subseteq \mathcal{D}^{pb}(A)$. It turns out that applying Lemma 0.1 and the fact that every SM-method is decomposable over $\mathcal{D}^b(A)$ (Section1.4) one can prove

2.1.1. Theorem. *For any $A \in \mathcal{R}^{\leftarrow b}(\mathcal{M})$,*

$$\mathcal{D}^{pb}(A; \mathcal{M}^2) = \mathcal{D}^{pb}(A).$$

This theorem implies

2.1.2. Corollary. *If $A \in \mathcal{R}^{\leftarrow b}(\mathcal{M})$, then the following statements hold:*

(1) *A is b-perfect in $\mathcal{M}^2$ if and only if it is b-perfect in $\mathcal{M}$;*
(2) *A is b-perfectly inconsistent in $\mathcal{M}^2$ if and only if it is b-perfectly inconsistent in $\mathcal{M}$.*

It turns out that statement (1) of Corollary 2.1.2 can be proved in the following stronger form

2.1.3. Proposition. *Let A be an arbitrary regular method. Then A is b-perfect in $\mathcal{M}^2$ if and only if it is b-perfect in $\mathcal{M}$. In particular, every b-perfect method in $\mathcal{M}^2$ has a b-majorant in $\mathcal{M}$.*

We need the well-known

2.1.4. Lemma. (see [2] and [3]) *Every regular matrix method is b-perfect in $\mathcal{M}$.*

From this lemma and Corollary 2.1.2 we get

2.1.5. Corollary. *Every regular matrix method is b-perfect in $\mathcal{M}^2$. A regular matrix method is b-perfectly inconsistent in $\mathcal{M}^2$ if and only if it is b-equivalent to the standard convergence.*

2. $\mathcal{D}^b$-majorants of regular SM-methods in the class $\mathcal{M}$.

Let $A \in \mathcal{M}^2$. Since A is decomposable over its bounded convergence field, we conclude that for any n such that $\chi(A^n) \neq 0$ the method $\chi(A^n)^{-1}A^{n0}$ is a $\mathcal{D}^b$-majorant of A in $\mathcal{M}$. Thus, a necessary condition for A that it has no $\mathcal{D}^b$-majorant in $\mathcal{M}$ is that $\chi(A^n) = 0$ for all n, however, this condition is not sufficient. For example, if B is a regular matrix method, then we have $B^{\uparrow} \in \mathcal{M}^2$ and $\chi(B^{\uparrow n}) = 0$ for all n but $B^{\uparrow} \simeq B \in \mathcal{M}$.

We shall regard the set T_b as a complex algebra under the coordinatewise operations. One can prove

2.2.1. Lemma. *There exist a method $A \in \mathcal{M}^2$ and $x_0 \in \mathcal{D}^b(A)$ such that the following conditions hold:*

(i) *x_0 is an invertible element of the algebra T_b and $A(x_0) = 1$;*
(ii) *A has a $\mathcal{D}^b$-majorant in $\mathcal{M}$;*
(iii) *if B is a $\mathcal{D}^b$-majorant of A in $\mathcal{M}$., then $B(x_0) = 0$.*

Applying this lemma (condition (ii) may be omitted) one can show

2.2.2. Theorem. *There exists a method $A \in \mathcal{M}^2$ which has no $\mathcal{D}^b$-majorant in $\mathcal{M}$.*

Clearly, every method satisfying Theorem 2.2.2 has no b-majorant, $\mathcal{D}$-majorant and majorant in $\mathcal{M}$, simultaneously. In particular, from

Proposition 2.1.3 it follows that such a method is not b-perfect in $\mathcal{M}^2$.

2.3. Perfect part of bounded convergence field.

A set $S \subseteq T$ is said to be a *bounded Toeplitz field* if there is a matrix method Λ such that $\mathcal{D}^b(\Lambda) = S$. One can prove

2.3.1. Theorem. *Every regular SM-method has a greatest b-minorant in $\mathcal{M}$, which means that $\mathcal{M}^{-b}(A) \neq \emptyset$ for $A \in \mathcal{M}^2$. Moreover, if $\Lambda \in \mathcal{M}^{-b}(A)$, then $\mathcal{D}^{pb}(A) = \mathcal{D}^b(\Lambda)$, and so, $\mathcal{D}^{pb}(A)$ is a bounded Toeplitz field.*

Applying this theorem and Lemma 2.1.4 we get the following corollaries.

2.3.2. Corollary. *Let $A = (\Lambda^n)$ be a regular SM-method defined by a sequence $\Lambda^1, \Lambda^2 \ldots$, of regular matrix methods. Then*

$$\mathcal{M}^{-b}(A) = \mathcal{M}^{-b}(\Lambda^1, \Lambda^2, \ldots).$$

2.3.3. Corollary. *For every finite or countable family $A^1, A^2, \ldots$ of regular SM-methods, there exists a greatest b-minorant in class $\mathcal{M}$. Moreover, if $\Lambda^i \in \mathcal{M}^{-b}(A^i)$ for $i = 1, 2, \ldots$, then*

$$\mathcal{M}^{-b}(A^1, A^2, \ldots) = \mathcal{M}^{-b}(\Lambda^1, \Lambda^2, \ldots).$$

In paper [6], by using a special construction, it is proved the existence of some increasing sequence of regular matrix methods that has no b-majorant in class $\mathcal{M}^2$. On the other hand, one can show that Theorem 2.3.1 implies

2.3.4. Corollary. *If $\Lambda^1 \subset \Lambda^2 \subset \ldots$ is an increasing sequence of regular matrix methods such that $\mathcal{M}^{\Rightarrow b}(\Lambda^1, \Lambda^2, \ldots) = \emptyset$, then $\mathcal{M}^{2 \Rightarrow b}(\Lambda^1, \Lambda^2, \ldots) = \emptyset$.*

The assumption of this corollary can be fulfilled since Brudno (see [2], Theorem 1 and the proof of Theorem 9) proved the existence of an increasing sequence $\Lambda_1 \subset \Lambda_2 \subset \dots$ of regular matrix methods such that there is no matrix method Γ satisfying $\Lambda_i \subset_b \Gamma$ for all i. Thus, for such a sequence there is no SM-method A or equivalently iteration product $\Theta[\Lambda]$ of matrix transformations such that $\Lambda_i \subset_b A$ or $\Lambda_i \subset_b \Theta[\Lambda]$ $(i = 1, 2, \dots)$. The latter observation for $\Theta[\Lambda]$ is exactly the main result proved in [6] (see Proposition 4.1) by applying some special considerations which are completely independent of those presented here.

As one knows if $\Lambda \in \mathcal{M}$, then $\mathcal{D}^b(\Lambda) = T_c$ or $\mathcal{D}^b(\Lambda)$ is a non-separable Banach subspace of T_b (compare [4], Theorem 8). Thus, from Theorem 2.3.1 we get

2.3.5. Corollary. *If $A \in \mathcal{M}^2$, then $\mathcal{D}^{pb}(A) = T_c$ or $\mathcal{D}^{pb}(A)$ is a non-separable Banach subspace of T_b.*

4. b-perfectly inconsistent SM-methods.

It is know (see [4], Theorem 10) that a regular matrix method is perfectly inconsistent if and only if every sequence limitable by it is convergent or unbounded. This property is not valid for regular SM-methods because there are regular SM-methods which limit some divergent sequences but no unbounded sequences. Moreover, such methods can be inconsistent and b-inconsistent, simultaneously. More precisely, one can prove

2.4.1. Proposition. *If A is a free regular SM-method limiting only bounded sequences, then A is inconsistent and b-inconsistent, simultaneously.*

It is easy to show that there are regular SM-methods A not b-equivalent to the standard convergence for which $\mathcal{D}^b(A)$ are separable subspaces of T_b. Hence and from Corollary 2.3.5 it follows that any such method is b-perfectly inconsistent. On the other hand, one can prove the following theorems.

2.4.2. Theorem. *There exists a method $A \in \mathcal{M}^2$ such that $\mathcal{D}^b(A)$ is a non-separable Banach subspace of T_b and $\mathcal{D}^{pb}(A) = T_c$.*

2.4.3. Theorem. *There exists a method $A \in \mathcal{M}^2$ such that $\mathcal{D}^b(A)$ is a non-separable and non-complete subspace of T_b and $\mathcal{D}^{pb}(A) = T_c$.*

2.5. Characterization of b-perfect SM-methods.

Applying Theorem 2.3.1 and Lemma 2.1.4 one can prove

2.5.1. Proposition. *Let $A \in \mathcal{M}^2$. Then the following conditions are equivalent:*

(a) *A has a least $\mathcal{D}^b$-majorant in $\mathcal{M}$;*
(b) *A is b-perfect;*
(c) *A is b-equivalent to a matrix method.*

As we know from the end of Section 1.3, there does not exist a matrix method b-equivalent to the almost convergence (the Lorentz Theorem). Hence and from Proposition 2.5.1, Corollary 1.3.3 and Theorem 2.1.1 we get

2.5.2. Corollary. *There is no regular SM-method b-equivalent to the almost convergence.*

If $A \in \mathcal{M}^2$, then by Lemma 0.1 we have $\mathcal{D}^{pb}(A) \subseteq \mathcal{D}_0^{pb}(A)$. It turns out that this inclusion can be proper. Namely, Lemma 2.2.1 immediately implies

2.5.3. Theorem. *There exists a method $A \in \mathcal{M}^2$ such that $\mathcal{D}^{pb}(A) \subset \mathcal{D}_0^{pb}(A)$.*

One can prove

2.5.4. Theorem. *A regular SM-method A is b-perfect if and only if $\mathcal{D}^b(A) = \mathcal{D}_0^{pb}(A)$.*

Clearly, the last theorem implies

2.5.5. Corollary. *A regular SM-method is not b-perfect if and only if there exists a method $B \in \mathcal{M}_0$ such that $\mathcal{D}^b(A) \subseteq \mathcal{D}^b(B)$ and $B(x_0) \neq 0$ for some $x_0 \in \mathcal{D}^b(A)$.*

2.6. Quasi b-order for SM-methods and matrix methods.

Applying Proposition 2.5.1 and Corollary 2.5.5 one can prove

2.6.1. Theorem. *Let A and B be regular SM-methods such that $A \subset_b B$ and $\mathcal{D}^{pb}(A) \subset \mathcal{D}^{pb}(B)$. Then there exists a regular SM-method C such that $A \subset_b C \subset_b B$, $\mathcal{D}^{pb}(A) \subset \mathcal{D}^{pb}(C) \subset \mathcal{D}^{pb}(B)$ and C is not b-equivalent to any matrix method.*

It is easy to show that under the assumptions of this theorem one can find A and B such that there is no regular matrix method Λ satisfying $A \subset_b \Lambda \subset_b$B. Clearly, Theorem 2.6.1 implies

2.6.2. Corollary. *Let Λ and Θ are regular matrix method such that $\Lambda \subset_b \Theta$. Then there exists a regular SM-method C such that $\Lambda \subset_b C \subset_b \Theta$ and C is not b-equivalent to any matrix method.*

The following theorem is well know (see [5], Theorem 4.3.3).

2.6.3. Theorem. *If (Λ_n) is a sequence of regular matrix methods such that $\Lambda_{n+1} \subset_b \Lambda_n$ for all n, then there exists $x_0 \in T_b \backslash T_c$ such that $x_0 \in \mathcal{D}^b(\Lambda_n)$ for all n.*

On the other hand, one can prove

2.6.4. Theorem. *There exists a sequence (A_n) of regular SM-methods satisfying the following conditions:*

(i) $A_{n+1} \subset_b A_n$ *for all* n,

(ii) $\bigcap_{n=1}^{\infty} \mathcal{D}(A_n) = T_c$.

3. Perfectness for full convergence fields

3.1. Topology for full convergence fields.

Let $\Lambda = (\lambda_i^m)$ be a matrix method. Following Mazur and Orlicz [4] we shall regard $\mathcal{D}(\Lambda)$ to be a B_0-space, i.e. a separable complete metrizable locally convex space, under the coordinatewise operations and the topology defined by the following system of pseudonorms:

$$\|x\|_i^1 = |x^i| \quad (i = 1, 2, \dots);$$

$$\|x\|_m^2 = \sup_k \left| \sum_{i=1}^{k} \lambda_i^m x^i \right| \quad (m = 1, 2, \dots);$$

$$\|x\|^3 = \sup_m \sum_i \left| \lambda_i^m x^i \right|.$$

It is known that $\mathcal{D}(\Lambda)$ is an Fk-space, which means that it is a Fréchet space such that the condition $x_n \to x_o$ in $\mathcal{D}(\Lambda)$ involves $x_n^i \to x_o^i$ for each i.

Let Λ and Θ be matrix methods. From the closed-graph Theorem and since $\mathcal{D}(\Lambda)$ and $\mathcal{D}(\Theta)$ are Fk-spaces, it follows that the inclusion map $i : \mathcal{D}(\Lambda) \hookrightarrow \mathcal{D}(\Theta)$ is continuous. This implies that for any $\Lambda \in \mathcal{M}$ we have $\mathrm{cl}_\Lambda T_c \subseteq \mathcal{D}^p(\Lambda; \mathcal{M})$ where $\mathrm{cl}_\Lambda T_c$ denotes the closure of T_c in the topology of $\mathcal{D}(\Lambda)$. On the other hand, since for $\Lambda, \Theta \in \mathcal{M}$ we have $\Lambda \subseteq_b \Theta$ provided that $\Lambda \subseteq \Theta$, it follows from Lemma 2.1.4 that $\mathcal{D}^b(\Lambda) \subseteq \mathcal{D}^p(\Lambda; \mathcal{M})$. More generally, it is known that $\mathrm{cl}_\Lambda T_c = \mathcal{D}^p(\Lambda; \mathcal{M})$ (see [4], Theorem 4 and [9], Theorem 6.3), which implies that $\mathrm{cl}_\Lambda \mathcal{D}^b(\Lambda) = \mathcal{D}^p(\Lambda; \mathcal{M})$.

Given an SM-method $A = (a_i^{nk})$, in the linear space $\mathcal{D}(A)$ we introduce a locally convex topology defined by the following system of pseudonorms:

$$\|x\|_i^1 = |x^i| \quad (i = 1, 2, \dots);$$

$$\|x\|_{nk}^2 = \sup_m \left| \sum_{i=1}^{m} a_i^{nk} x^i \right| \quad (n = 1, 2, \dots;\ k = 1, 2, \dots);$$

$$\|x\|_n^3 = \sup_k \left| \sum_i a_i^{nk} x^i \right| \quad (n = 1, 2, \dots);$$

$$\|x\|^4 = \sup_n \left| \lim_k \sum_i a_i^{nk} x^i \right|.$$

We shall regard $\mathcal{D}_0(A)$ as a topological linear subspace of $\mathcal{D}(A)$. Clearly, $\mathcal{D}(A)$ and $\mathcal{D}_0(A)$ are B_0-spaces and separable Fk-spaces. Analogously as for matrices, we get

3.1.1. Lemma. *Let A and B be SM-methods such that $\mathcal{D}(A) \subseteq \mathcal{D}(B)$. Then the inclusion map $i : \mathcal{D}(A) \hookrightarrow \mathcal{D}(B)$ is continuous.*

Note that if Λ is a matrix method, then the topology of $\mathcal{D}(\Lambda^{\uparrow})$ ($\mathcal{D}(\Lambda^{\uparrow}) = \mathcal{D}(\Lambda)$) coincides with that of $\mathcal{D}(\Lambda)$.

Let now $\Lambda = (\lambda_i^m)$ and $\Theta = (\theta_m^n)$ be matrices. Since the methods $\Theta[\Lambda]$ and $\Theta\#\Lambda$ are equivalent by Proposition 1.1.3, we have $\mathcal{D}(\Theta[\Lambda]) = \mathcal{D}(\Theta\#\Lambda)$ and the locally convex topology in $\mathcal{D}(\Theta[\Lambda])$ is defined to be such topology in $\mathcal{D}(\Theta\#\Lambda)$, which means that it is given by the following system of pseudonorms:

$$\|x\|_i^1 = |x^i| \quad (i = 1, 2, \dots);$$

$$\|x\|_m^2 = \sup_l \left| \sum_{i=1}^{l} \lambda_i^m x^i \right| \quad (m = 1, 2, \dots);$$

$$\|x\|_{nk}^3 = \sup_l \left| \sum_{m=1}^{k} \theta_m^n \sum_{i=1}^{l} \lambda_i^m x^i \right| \quad (n = 1, 2, \dots;\ k = 1, 2, \dots);$$

$$\|x\|_n^4 = \sup_k \left| \sum_{m=1}^{k} \theta_m^n \sum_i \lambda_i x^i \right| \quad (n = 1, 2, \dots);$$

$$\|x\|^5 = \sup_n \left| \sum_m \theta_m^n \sum_i \lambda_i^m x^i \right|.$$

If A is an SM-method, then by a linear functional f on $\mathcal{D}(A)$ we shall mean an algebraically $\mathbb{C}$-linear functional $f : \mathcal{D}(A) \to \mathbb{C}$ continuous in the topology of $\mathcal{D}(A)$. Applying the general form of such a functional one can prove

3.1.2. Lemma. *Let A be a convergence zero-preserving SM-method. If f is a linear functional on $\mathcal{D}(A)$, then the following decomposition holds:*

$$\text{(D1)}\quad f(x) = \sum_i \alpha_i x^i + \sum_{n=1}^{l} \sum_k \beta_{nk} A^{nk}(x) + \sum_n \gamma_n A^n(x) + \delta A(x)$$

where $\sum_i |\alpha_i| < \infty$, $\sum_k |\beta_{nk}| < \infty$ $(n = 1, 2, \dots, l)$ and $\sum_n |\gamma_n| < \infty$. Moreover, there exists an SM-method B such that $\mathcal{D}(A) \subseteq \mathcal{D}(B)$ and

$B(\cdot)|\mathcal{D}(A) = f$. Remark that such a method B can be defined by the method A and the coefficients from decomposition $(D1)$.

It turns out that if we omit the assumption of this lemma that A is convergence zero-preserving, then in general a linear functional on $\mathcal{D}(A)$ may be only expressed in a form much more complicated than $(D1)$. Clearly, we can reformulate Lemma 3.1.2 for an IPM-method $\Theta[\Lambda]$ where $\Theta = (\theta^n_m)$ and $\Lambda = (\lambda^m_i)$. In this case decomposition $(D1)$ takes the following form:

$$\text{(D2)} \qquad \begin{aligned} f(x) = \sum_i a_i\, x^i + \sum_{n=1}^{l} \sum_m b^m_n\, \theta^n_m \sum_i \lambda^m_i\, x^i \\ + \sum_n c_n \sum_m \theta^n_m \sum_i \lambda^m_i\, x^i + d \lim_n \sum_m \theta^n_m \sum_i \lambda^m_i\, x^i \end{aligned}$$

where $\sum_i |a_i| < \infty$, $\sum_m |b^m_n - b^{m+1}_n| < \infty$ $(n = 1, 2, ..., l)$ and $\sum_n |c_n| < \infty$. In particular, by putting Θ to be the identity matrix we get the following well-known decomposition of a linear functional f on $\mathcal{D}(\Lambda)$ (see [9], Theorem 5.2):

$$\text{(D3)} \qquad f(x) = \sum_i a_i\, x^i + \sum_n b_n\, \Lambda^n(x) + c\, \Lambda(x)$$

where $\sum_i |a_i| < \infty$, $\sum_n |b_n| < \infty$. One can see that for a matrix Λ, by putting Λ to be the identity matrix in $(D2)$ or by putting Λ in (D1) instead of A, we also get an analogous decomposition to (D3). In this case the assumption that the methods Θ and Λ are convergence zero-preserving plays an essential role.

3.2. Perfectness in the classes $\mathcal{M}^2$ and $\mathcal{M}$.

Let A be an SM-method. If $S \subseteq \mathcal{D}(A)$, then by $\mathrm{cl}_A\, S$ will be denoted the closure of S in the topology of $\mathcal{D}(A)$. One can see that Lemmas 3.1.1 and 3.1.2 imply

3.2.1. Lemma. *Let A be a convergence zero-preserving SM-method and let S be a linear subspace of $\mathcal{D}(A)$. Then $x \in \mathrm{cl}_A\, S$ if and only if* *for any SM-method B such that $\mathcal{D}(A) \subseteq \mathcal{D}(B)$ and $B(\cdot)|S| = A(\cdot)|S|$ we have $B(x) = A(x)$.*

This lemma immediately implies

3.2.2. Theorem. *If $A \in \mathcal{M}^2$, then $\mathcal{D}^p(A; \mathcal{M}^2) = \mathrm{cl}_A T_c$.*

Since for any matrix method Λ we have $\mathcal{D}^p(\Lambda) = \mathrm{cl}_\Lambda T_c$, this theorem implies the following corollaries.

3.2.3. Corollary. *If $\Lambda \in \mathcal{M}$, then $\mathcal{D}^p(\Lambda; \mathcal{M}^2) = \mathcal{D}^p(\Lambda)$.*

3.2.4. Corollary. *Let $\Lambda \in \mathcal{M}$. Then Λ is perfect in $\mathcal{M}^2$ if and only if it is perfect in $\mathcal{M}$.*

One can prove

3.2.5. Lemma. *Let $\Lambda_1 \supseteq \Lambda_2 \supseteq \ldots$ be a decreasing sequence of perfect regular matrix methods. Then the SM-method (Λ_n) is perfect in $\mathcal{M}^2$.*

From this lemma and some other considerations one can derive

3.2.6. Theorem. *There exists a simple regular SM-method which is perfect in $\mathcal{M}^2$ but is not equivalent to any matrix method.*

This is the first fact indicating a difference between the perfectness and the b-perfectness for SM-methods (compare Proposition 2.5.1 and Corollary 2.1.2).

3.3. Iteration products of regular matrix transformations.

If Λ is a regular matrix method, we set

$$R(\Lambda) = \{\Lambda^\sim(x) : x \in D^\sim(\Lambda)\} \quad \text{and}$$
$$R^p(\Lambda) = \{\Lambda^\sim(x) : x \in D^p(\Lambda)\}.$$

One can prove

3.3.1. Theorem. *Let Λ and Θ be regular matrix methods satisfying the following conditions:*

(i) *$R(\Lambda) \cap \mathcal{D}(\Theta)$ is a closed subspace of $\mathcal{D}(\Theta)$;*
(ii) *if $x \in \mathcal{D}^\sim(\Lambda)$ and $\Lambda^\sim(x) \in R^p(\Lambda)$, then $x \in \mathcal{D}^p(\Lambda)$.*

Then $\mathcal{D}^p(\Theta[\Lambda];\mathcal{M}^2) = \{x \in \mathcal{D}(\Theta[\Lambda]) : \Lambda^\sim(x) \in \mathrm{cl}_\Theta R^p(\Lambda)\}$.

This theorem implies the following corollaries.

3.3.2. Corollary. *Let* Λ *and* Θ *be regular matrix methods. If* $R(\Lambda) = T$ *and* Θ *is non-perfect, then* $\Theta[\Lambda]$ *is non-perfect in* $\mathcal{M}^2$.

3.3.3. Corollary. *If* Λ *and* Θ *are perfect regular matrix methods such that* $R(\Lambda) = T$, *then* $\Theta[\Lambda]$ *is perfect in* $\mathcal{M}^2$.

It is seen that if Λ, Θ_1 and Θ_2 are matrix methods such that $\Theta_1 \simeq \Theta_2$, then $\Theta_1[\Lambda] \simeq \Theta_2[\Lambda]$. In particular, $\Theta_1[\Lambda]$ is perfect in $\mathcal{M}^2$ if and only if so is $\Theta_2[\Lambda]$. It turns out that if Λ_1, Λ_2 and Θ are matrix methods such that $\Lambda_1 \simeq \Lambda_2$, then the methods $\Theta[\Lambda_1]$ and $\Theta[\Lambda_2]$ may be not equivalent in general. More precisely, we have

3.3.4. Theorem. *There exist matrix methods* $\Lambda_1, \Lambda_2, \Theta \in \mathcal{M}$ *with finite (infinite) rows satisfying the following conditions:*

(i) Λ_1, Λ_2 *and* Θ *are perfect;*
(ii) $\Lambda_1 \simeq \Lambda_2$;
(iii) $\Theta[\Lambda_1]$ *is perfect in* $\mathcal{M}^2$;
(iv) $\Theta[\Lambda_2]$ *is non-perfect in* $\mathcal{M}^2$.

Since the methods $\Theta[\Lambda]$ and $\Theta\Lambda$ are equivalent provided that Θ has finite rows (Proposition 1.1.5), Theorem 3.3.4 implies

3.3.5. Corollary. *There exist matrix methods* $\Lambda, \Theta \in \mathcal{M}$ *with finite rows such that the methods* Λ *and* Θ *are perfect but the method* $\Lambda\Theta$ *is non-perfect.*

One can prove

3.3.6. Theorem. *There exist matrix methods* $\Lambda_1, \Lambda_2, \Theta \in \mathcal{M}$ *with finite (infinite) rows satisfying the following conditions:*

(i) Λ_1, Λ_2 *and* Θ *are non-perfect;*
(ii) $\Lambda_1 \simeq \Lambda_2$;
(iii) $\Theta[\Lambda_1]$ *is perfect in* $\mathcal{M}^2$;
(iv) $\Theta[\Lambda_2]$ *is non-perfect in* $\mathcal{M}^2$.

Clearly, this theorem implies

3.3.7. Corollary. *There exist matrices Λ, $\Theta \in \mathcal{M}$ with finite rows such that the methods Λ and Θ are non-perfect but the method $\Lambda\Theta$ is perfect.*

3.4. Reducible methods.

We shall denote by $\mathcal{M}_r^2$ the class of all reducible regular SM-methods. If $S \subseteq T$, then by $\mathcal{M}_r^2(S)$ will be denoted the class of all regular SM-methods which are reducible over S. Since every method from $\mathcal{M}_r^2$ has a decomposable SM-representation, we may expect that properties of such methods which are connected with the perfectness should be analogous to those of methods from $\mathcal{M}^2$ which are connected with the b-perfectness.

One can prove (compare Theorem 2.1.1)

3.4.1. Theorem. *For any $A \in \mathcal{R}^{\leftarrow}(\mathcal{M})$,*

$$\mathcal{D}^p(A;\ \mathcal{M}_r^2(\mathcal{D}(A))) = \mathcal{D}^p(A;\ \mathcal{M}_r^2) = \mathcal{D}^p(A).$$

This theorem implies

3.4.2. Corollary. *If $A \in \mathcal{R}^{\leftarrow}(\mathcal{M})$, then the following statements hold:*

(1) *A is perfect in $\mathcal{M}_r^2$ if and only if it is perfect in $\mathcal{M}$;*
(2) *A is perfectly inconsistent in $\mathcal{M}_r^2$ if and only if it is perfectly inconsistent in $\mathcal{M}$.*

Applying decomposition $(D1)$ from Section 3.1 one has

3.4.2. Lemma. *If $A \in \mathcal{M}_r^2$, $B \in \mathcal{M}^2$ and $\mathcal{D}(A) \subseteq \mathcal{D}(B)$, then there exists a method $C \in \mathcal{M}_r^2(\mathcal{D}(A))$ such that $\mathcal{D}(A) \subseteq \mathcal{D}(C)$ and both B and C are consistent over $\mathcal{D}(A)$.*

This lemma immediately implies

3.4.3. Corollary. *For any $A \in \mathcal{M}_r^2$,*

$$\mathcal{D}^p(A;\ \mathcal{M}^2) = \mathcal{D}^p(A;\ \mathcal{M}_r^2).$$

Clearly, from Theorem 3.4.1 and Corollary 3.4.3 we get

3.4.4. Corollary. *Let $A \in \mathcal{M}_r^{2\leftarrow}(\mathcal{M})$. Then the following conditions are equivalent:*

(a) *A is perfect in $\mathcal{M}^2$;*
(b) *A is perfect in $\mathcal{M}_r^2(\mathcal{D}(A))$;*
(c) *A is perfect in $\mathcal{M}_r^2$;*
(d) *A is perfect in $\mathcal{M}$.*

Denote by $\mathcal{M}_p^2$ the class of all regular SM-methods which have simple SM-representations perfect in $\mathcal{M}^2$. It follows from Theorem 3.2.6 that a method from $\mathcal{M}_p^2$ need not be equivalent to any matrix method. A set $S \subseteq T$ is said to be a *perfect simple field* if there exists $A \in \mathcal{M}_p^2$ such that $\mathcal{D}(A) = S$.

One can prove

3.4.5. Theorem. *If $A \in \mathcal{M}_r^2$, then $\mathcal{D}^p(A; \mathcal{M}^2)$ is a perfect simple field.*

Clearly, this theorem and Corollary 3.4.3 imply

3.4.6. Corollary. *$\mathcal{M}_p^2$ is exactly the class of all reducible regular SM-methods perfect in $\mathcal{M}^2$ or equivalently in $\mathcal{M}_r^2$.*

We shall regard the set T as a complex algebra under the coordinatewise operations. Analogously as Lemma 2.2.1 one can prove

3.4.7. Lemma. *There exist a method $A \in \mathcal{M}_r^2$ and $x_0 \in \mathcal{D}(A)$ such that the following conditions hold:*

(i) *x_0 is an invertible element of the algebra T and $A(x_0) = 1$;*
(ii) *A has a $\mathcal{D}$-majorant in $\mathcal{M}$;*
(iii) *if B is a $\mathcal{D}$-majorant of A in $\mathcal{M}_\bullet$, then $B(x_0) = 0$.*

This lemma implies the following theorems.

3.4.8. Theorem. *There exists a method $A \in \mathcal{M}_r^2$ which has no $\mathcal{D}$-majorant in $\mathcal{M}$.*

3.4.9. Theorem. *There exists a method $A \in \mathcal{M}_r^{2\leftarrow}(\mathcal{M})$ such that $\mathcal{D}^p(A) \subset \mathcal{D}_0^p(A)$.*

It is seen that every method A satisfying the latter theorem cannot be equivalent to any matrix method.

Analogously as Theorem 2.5.4 one can prove

3.4.10. Theorem. *Let $A \in \mathcal{M}_r^{2\leftarrow}(\mathcal{M})$. Then A is perfect in $\mathcal{M}^2$ if and only if $\mathcal{D}(A) = \mathcal{D}_0^p(A)$.*

This theorem immediately implies

3.4.11. Corollary. *A method $A \in \mathcal{M}_r^{2\leftarrow}(\mathcal{M})$ is non-perfect in $\mathcal{M}$ if and only if there exists a $\mathcal{D}$-majorant B of A in $\mathcal{M}_0$ such that $B(x_0) \neq 0$ for some $x_0 \in \mathcal{D}(A)$.*

3.5. Irreducible methods.

This section contains some existence results on irreducible methods with respect to the perfectness. These results can be proved by using special constructions. We present their list in which corollaries follow from the preceding theorems.

3.5.1. Theorem. *There exists an irreducible method from $\mathcal{M}^2$ which is perfect in $\mathcal{M}^2$ and has no $\mathcal{D}$-majorant in $\mathcal{M}$.*

3.5.2. Corollary. *There exists an irreducible method from $\mathcal{M}^2$ which is perfect in $\mathcal{M}^2$ and has no $\mathcal{D}$-majorant in $\mathcal{M}_p^2$.*

3.5.3. Theorem. *There exists an irreducible method from $\mathcal{M}^2$ which is non-perfect in $\mathcal{M}^2$ and has no $\mathcal{D}$-majorant in $\mathcal{M}_r^2$.*

3.5.4. Corollary. *There exists an irreducible method from $\mathcal{M}^2$ which is perfect (non-perfect) in $\mathcal{M}^2$ such that every its SM-representation is strongly indecomposable.*

3.5.5. Theorem. *There exists an irreducible method from $\mathcal{M}^2$ which is perfect in the classes $\mathcal{M}^2$ and $\mathcal{M}$.*

3.5.6. Theorem. *There exists an irreducible method from $\mathcal{M}^2$ which has a majorant in $\mathcal{M}$ and is non-perfect in the classes $\mathcal{M}^2$ and $\mathcal{M}$.*

3.5.7. Theorem. *There exists an irreducible method from $\mathcal{M}^2$ which is non-perfect in $\mathcal{M}^2$ but is perfect in $\mathcal{M}$.*

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O METODACH SUMOWALNOŚCI OKREŚLONYCH PRZEZ CIĄGOWE METODY MACIERZOWE ORAZ OKREŚLONYCH PRZEZ ZŁOŻENIE PRZEKSZTAŁCEŃ MACIERZOWYCH

Praca jest streszczeniem opartym głównie na mojej pracy doktorskiej z uwzględnieniem pewnych modyfikacji i uzupełnień . Na początku, po omówieniu ogólnego pojęcia metody w sensie teorii sumowalności, wprowadzono pojęcie ciągowej metody macierzowej (sequential matrix method, SM-method). Dalej, zwrócono uwagę na równoważność takich metod z dobrze znanymi metodami określonymi przez złożenie przekształceń macierzowych, która pozwala badać te ostatnie metody przez sprowadzenie ich do ciągowych metod macierzowych. W szczególności, dotyczy to własności metod związanych z regularnością i prawie regularnością (Paragrafy 1.2 i 1.3) oraz związanych z opisem topologii w pełnym polu metody (Paragraf 3.1). Celem głównym pracy jest przedstawienie wyników w zakresie b-doskonałości i doskonałości regularnych ciągowych metod macierzowych (Rozdziały 2 i 3) z zastosowaniem do opisu odpowiednich własności metod określonych przez złożenie przekształceń macierzowych.

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