A C T A U N I V E R S I T A T I S L O D Z I E N S I S
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ON GENERAL ESTIMATIONS OF COEFFICIENTS OF BOUNDED SYMMETRIC UNIVALENT FUNCTIONS

Let $S_R(M)$, $M > 1$, be the class of functions

$$F(z) = z + \sum_{n=2}^{\infty} A_{nF} z^n$$

holomorphic, univalent and bounded by a constant M in the unit disc E. If K is an odd positive integer, wherаes N-even, and λ, μ are real numbers such that $\lambda \geqslant 0$ and $\mu > 0$, then there exists a constant $M_o, M_o > 1$, such that, for all $M > M_o$ in the class $S_R(M)$, the inequality

$$A_{KF} + A_{NF} \leqslant P_{K,M} + P_{N,M}$$

takes place, where

$$w = P_M(z) = z + \sum_{n=2}^{\infty} P_{n,M} z^n, \ z \in E,$$

is a Pick function given by the equation

$$\frac{w}{\left(1 - \frac{w}{M}\right)^2} = \frac{z}{(1 - z)^2}, \quad z \in E.$$

1. INTRODUCTION

Let S be the class of functions

$$(1) \qquad F(z) = z + \sum_{n=2}^{\infty} A_{nF} z^n$$

holomorphic and univalent in the disc $E = \{z : |z| < 1\}$, and $S(M)$, $M > 1$, the subclass of the above class, consisting of functions bounded by M, that is, of those which satisfy the condition

$$|F(z)| \leqslant M, \quad z \in E.$$

Charzyński and Tammi set the following hypothesis for the classes $S(M)$: for every $N = 2,3,4, \ldots,$ there exists a constant $M_N > 1$ such that, for all $M < M_N$ and every function $F \in S(M)$, the sharp estimation

$$|A_{NF}| \leqslant P_{N,M}^{(N-1)}$$

takes place, where

$$P_{N,M}^{(N-1)} = \frac{2}{N-1}\left(1 - \frac{1}{M^{N-1}}\right)$$

is the N-th coefficient of Taylor expansion (1) of the Pick function $w = P_M^{(N-1)}(z)$ (symmetric, of order N-1) given by the equation

$$\frac{w}{\left[1 - \left(\frac{w}{M}\right)^{N-1}\right]^{\frac{2}{N-1}}} = \frac{z}{\left[1 - z^{N-1}\right]^{\frac{2}{N-1}}}, \quad z \in E,$$

and satisfying the condition $P_M^{(N-1)}(0) = 0$.

This hypothesis was positively determined by S i e w i e r s k i ([13], [14], [15]) and, in some other way, by S c h i f f e r and T a m m i [12].

Jakubowski raised for the classes $S(M)$ a hypothesis antipodal to the above-mentioned one: for every even $N = 2, 4, 6, \ldots$, there exists a constant $M_N > 1$ such that, for all $M > M_N$ and every function $F \in S(M)$, the sharp estimation

$$|A_{NF}| \leqslant P_{N,M}$$

takes place, where

$$P_{N,M} = \tag{2}$$

$$= N + \sum_{m=2}^{N}\left[(-1)^{m+1}\,\frac{2^m}{M^{m-1}}\cdot\frac{1\cdot 3\cdot\ldots\cdot(2m-1)}{(m+1)!}\sum_{\substack{(s_1,\ldots,s_m)\\ s_1+\ldots+s_m=N\\ 1\leqslant s_j\leqslant N,\, j=1,\ldots,m}} s_1 s_2\ldots s_m\right]$$

(cf. [4]) is the N-th coefficient of Taylor expansion (1) of the Pick function $w = P_M(z)$ (symmetric, of order 1) given by the equation

$$\frac{w}{\left(1-\frac{w}{M}\right)^2} = \frac{z}{(1-z)^2}, \quad z \in E, \tag{3}$$

and satisfying the condition $P_M(0) = 0$.

The premises for raising this hypothesis were the estimations in the classes $S(M)$, known earlier ([7], [10]):

$$|A_{2F}| \leqslant P_{2,M} \quad \text{if} \quad M > 1,$$

$$|A_{4F}| \leqslant P_{4,M} \quad \text{if} \quad M > 700,$$

on whose grounds, as can be seen, one may adopt, for instance, $M_2 = 1$ and $M_4 = 700$.

However, for any even N, the hypothesis has not been determined till now.

Note that, for any odd N, the above hypothesis is not valid since, as early as $N = 3$, in the class $S(M)$ the sharp estimation

$$|A_{3F}| \leqslant 1 + 2\lambda^2 - 4\lambda M^{-1} + M^{-2} \quad \text{for} \quad e \leqslant M < +\infty \tag{4}$$

holds, where λ is the greater root of equation $\lambda \log \lambda = -M^{-1}$; the third coefficient of the Pick function $w = P_M(z)$ is, as can easily be verified, less than the right-hand side of (4).

Denote by S_R and $S_R(M)$, $M > 1$, the subclasses of, respectively, S and $S(M)$ of functions with real coefficients.

Jakubowski raised for the classes $S_R(M)$ a hypothesis analogous to the previous one: for every even $N = 2,4,6, \ldots,$ there exists a constant $M_N > 1$ such that, for all $M > M_N$ and every function $F \in S_R(M)$, the sharp estimation

$$A_{NF} \leqslant P_{N,M} \tag{5}$$

takes place, where $P_{N,M}$ is, as previously, the N-th coefficient of Taylor expansion (1) of the Pick function $w = P_M(z)$ given by equation (3) and satisfying the condition $P_M(0) = 0$.

An additional premise for the supposition and possibilities of a positive solution to the problem was the following result of Dieudonné ([1]): for every function $F \in S_R$,

$$A_{nF} \leqslant P_{n,\infty}, \quad n = 2,3,4, \ldots, \tag{6}$$

where $P_{n,\infty} = n$ is the n-th coefficient in Taylor expansion (1) of the Koebe function

$$\mathcal{K}(z) = P_{\infty}(z) = \frac{z}{(1-z)^2}, \quad z \in E, \tag{7}$$

being the limit case of the Pick function $w = P_M(z)$ as one passes in equation (3) with M to infinity. Moreover, Koebe function (7) is the only function for which equality in estimation (6) holds when n is even.

The use of the above fact, the differential-functional equation of extremal functions and the theory of Γ-structures allowed to determine Jakubowski's hypothesis positively in the class $S_R(M)$ ([4], [5], [16], [17]).

The present paper constitutes a generalization of the above result. Namely, in the class $S_R(M)$, $M > 1$, instead of single coefficients we consider some of their linear combinations of type $\lambda A_{KF} + \mu A_{NF}$, where K and N are any positive integers,

λ, μ - any non-negative real numbers. In virtue of estimation (5), it is evident that, if we assume K and N to be even, then, for M sufficiently large, the maximum of $\lambda A_{KF} + \mu A_{NF}$ is realized by the Pick function $w = P_M(z)$ only. Consequently, non-trivial is the case when K,N are any positive integers, N - even, K - odd. Besides, it can be seen from estimation (4) that the coefficients λ, μ cannot be arbitrary; in this context, we shall further assume that $\lambda \geqslant 0$ and $\mu > 0$.

The method used in the paper allows one to avoid complicated integration of the differential-functional equation of extremal functions (e.g., [3], [6], [11]); instead, one makes use of the theory of Γ-structures and the above-mentioned result of Dieudonné, including the onliness of the Koebe function in estimation (6) for n even.

2. THE FUNCTIONAL AND AUXILIARY RESULTS

Consider a real functional

$$J(F) = \lambda A_{KF} + \mu A_{NF}, \quad F \in S_R(M), \tag{8}$$

where K,N are any positive integers, K - odd, N - even; λ, μ - any real numbers, $\lambda \geqslant 0$, $\mu > 0$.

It follows from the Weierstrass theorem that functional (8) is continuous, whereas the family $S_R(M)$ is compact in the topology of almost uniform convergence. Consequently, for every $M > 1$, in the family $S_R(M)$ there is at least one function realizing the maximum of functional (8). In the sequel, each function F_0 for which

$$\max_{F \in S_R(M)} J(F) = J(F_0)$$

will be shortly called extremal function.

We shall now give some information on the Pick functions $w = P_M(z)$ given by equation (3) and satisfying the condition $P_M(0) = 0$.

First of all, note that each of the functions $P_M(z)$, $M > 1$, belongs to the class $S_R(M)$ since it can be represented in the form

$$P_M(z) = M\mathcal{K}^{-1}\left(\frac{1}{M}\mathcal{K}(z)\right), \quad z \in E,$$

where $\mathcal{K}$ is Koebe function (7). From this relation it also follows that every function $w = P_M(z) = \frac{1}{M}P_M(z)$, $M > 1$, maps the disc $|z| < 1$ onto the disc $|w| < 1$ cut along the radius from -1 to $r_M = -2M + 1 + 2\sqrt{M(M-1)}$.

Next, note that, in accordance with (2), the convergences

$$\lim_{M\to+\infty} \Gamma_{n,M} = n, \quad n = 2,3, \ldots, \tag{9}$$

hold.

For the extremal functions, the following property takes place:

Let $(M_h)_{h=1,2,\ldots}$ be any sequence of real numbers, $M_h > 1$, $h = 1,2, \ldots,$ such that $\lim_{h\to\infty} M_h = +\infty$, and let

$$(F_h(z))_{h=1,2,\ldots}, \quad z \in E,$$

be any sequence of extremal functions realizing the maximum of functional (8) in the respective classes $S_R(M_u)$, $h = 1,2, \ldots$ Then the sequence $(F_h(z))_{h=1,2,\ldots}$ is almost uniformly convergent in the disc E to the Koebe function $\mathcal{K}(z)$.

Indeed, denote

$$F_h(z) = z + \sum_{n=2}^{\infty} A_{nh} z^n, \quad h = 1,2, \ldots, \quad z \in E.$$

Since, for every h, $h = 1,2, \ldots,$ $P_{M_h}(z) \in S_R(M_h)$ and $F_h \in S_R$, therefore

$$\lambda P_{K,M_h} + \mu P_{N,M_h} \leqslant \lambda A_{Kh} + \mu A_{Nh} \leqslant \lambda K + \mu N.$$

Consequently, in view of (9), we get

$$\lim_{h\to+\infty} [\lambda A_{Kh} + \mu A_{Nh}] = \lambda K + \mu N. \tag{10}$$

Since the sequence $(F_h)_{h=1,2,\ldots}$ is a normal and almost commonly bounded sequence in the disc E, it suffices to prove that any subsequence of $(F_h)_{h=1,2,\ldots}$, almost uniformly convergent in E, converges to the function $\mathcal{K}$.

So, take any such subsequence $(F_j)_{j=1,2,\ldots}$, almost uniformly convergent in the disc E to some function $\tilde{F}$. It follows from the compactness of the class S_R that $\tilde{F} \in S_R$. From condition (10) and the Weierstrass theorem we conclude that

$$\lambda A_{K\tilde{F}} + \mu A_{N\tilde{F}} = \lambda K + \mu N,$$

which, in view of Dieudonné estimation (6), yields

$$A_{N\tilde{F}} = N, \tag{11}$$

and, since Koebe function (7) is the only one in the family S_R for which (11) holds, there must be that $\tilde{F} = \mathcal{K}$.

Note that from the above property of extremal functions follows immediately the almost uniform convergence in E of the sequence $(F_h^m(z))_{h=1,2,\ldots}$, $m = 2,3,\ldots$, of powers of extremal functions in the families $S_R(M_h)$, $h = 1,2,\ldots$, to the function $\mathcal{K}^m(z)$, where $\mathcal{K}$ is a Koebe function. In consequence, we shall obtain another property of extremal functions.

Let m be any positive integer, n - any index, $n = m, m + 1, \ldots$ For every number $\varepsilon > 0$, there exists a constant $M_\varepsilon > 1$ such that, for all $M > M_\varepsilon$ and every function F extremal in the class $S_R(M)$, where $M > M_\varepsilon$, the condition

$$|A_{nF}^{(m)} - A_{n\mathcal{K}}^{(m)}| < \varepsilon$$

is satisfied, with that the coefficients $A_{nF}^{(m)}$, $m = 2,3,\ldots$, $n = m, m+1, \ldots$, are given by the formula

$$F^m(z) = \sum_{n=m}^{\infty} A_{nF}^{(m)} z^z, \quad z \in E, \tag{12}$$

and $A_{nF}^{(1)} = A_{nF}$, $n = 2,3,\ldots$, $A_{1F}^{(1)} = 1$.

Really, otherwise, for any fixed m and $n = m, m+1, \ldots$, there exists a real number ε_0 such that, for every M_{ε_0}, one can find a constant M, $M > M_{\varepsilon_0}$, and a function F extremal in the class $S_R(M)$, $M > M_{\varepsilon_0}$, so that $|A_{nF}^{(m)} - A_{n\varkappa}^{(m)}| > \varepsilon_0$. Then there exist an increasing sequence $(M_h)_{h=1,2,\ldots}$ of real numbers $(\lim_{h\to\infty} M_h = +\infty)$ and its corresponding sequence $(F_h^m)_{h=1,2,\ldots}$ of powers of extremal functions in the classes $S_R(M_h)$, $h = 1,2,\ldots$, such that $|A_{nF_h}^{(m)} - A_{n\varkappa}^{(m)}| \geqslant \varepsilon_0$, which contradicts the almost uniform convergence on the sequence $(F_h^m)_{h=1,2,\ldots}$ to the function $\varkappa^m$ in the disc E.

3. PROOF OF THE FUNDAMENTAL THEOREM

We shall prove the following

Theorem. Let K, N be any fixed positive integers, K - odd, N - even; λ, μ - any real numbers, $\lambda \geqslant 0$, $\mu > 0$. Then there exists a constant M_0, $M_0 > 1$, such that, for all $M > M_0$ and every function $F \in S_R(M)$, the estimation

$$\lambda A_{KF} + \mu A_{NF} \leqslant \lambda P_{K,M} + \mu P_{N,M} \tag{13}$$

is true, where

$$w = P_M(z) = z + \sum_{n=2}^{\infty} P_{n,M} z^n$$

is a Pick function given by the equation

$$\frac{w}{\left(1 - \frac{w}{M}\right)^2} = \frac{z}{(1-z)^2}, \quad z \in E,$$

and satisfying the condition $P_M(0) = 0$. This funtion is the only one for which, with a given M, $M > M_0$, equality holds in estimation (13).

The proof of the theorem will consist of two parts.

3.1. The differential-functional equation for extremal functions

Without loss of generality, assume that $N < K$.

It is well known [2] that every function $w = f(z) = \frac{1}{M} F(z)$, where F is an extremal function in the family $S_R(M)$, $M > 1$, satisfies the following differential-funtional equation:

$$(14)\qquad \left(\frac{z\,w'}{w}\right)^2 \mathcal{M}(w) = \mathcal{N}(z), \quad 0 < |z| < 1,$$

where

$$(15)\qquad \mathcal{M}(w) = \lambda \sum_{m=2}^{K} \frac{A_{KF}^{(m)}}{M^{m-1}} \left(w^{m-1} + \frac{1}{w^{m-1}}\right) + \mu \sum_{m=2}^{N} \frac{A_{NF}^{(m)}}{M^{m-1}} \left(w^{m-1} + \frac{1}{w^{m-1}}\right) - \mathcal{P},$$

$$(16)\qquad \mathcal{N}(z) = \lambda(K-1)A_{KF} + \mu(N-1)A_{NF} + \lambda \sum_{m=2}^{K} (K-m+1)A_{K-m+1,F}\left(z^{m-1} + \frac{1}{z^{m-1}}\right) + \mu \sum_{m=2}^{N} (N-m+1)A_{N-m+1,F}\left(z^{m-1} + \frac{1}{z^{m-1}}\right) - \mathcal{P},$$

$$(17)\qquad \mathcal{P} = \min_{0 \leqslant x \leqslant 2\pi} \operatorname{Re}\left[\lambda \sum_{m=2}^{K} \frac{A_{KF}^{(m)}}{M^{m-1}} e^{ix(m-1)} + \mu \sum_{m=2}^{N} \frac{A_{NF}^{(m)}}{M^{m-1}} e^{ix(m-1)}\right],$$

the numbers $A_{nF}^{(m)}$, $n = 1, 2, \ldots$, $n = m$, $m + 1$, ..., are given by formula (12). The functions $\mathcal{M}(w)$ and $\mathcal{N}(z)$ assume, respectively, on the circles $|w| = 1$ and $|z| = 1$ real non-negative values. Either of these functions has on the respective circle at least one zero of even multiplicity. Let us still observe that, if $\mathcal{M}(w_o) = 0$, then $\mathcal{M}(\overline{w}_o) = 0$, $\mathcal{M}(\frac{1}{w_o}) = 0$ and $\mathcal{M}(\frac{1}{\overline{w}_o}) = 0$, and if $\mathcal{N}(z_o) = 0$, then also $\mathcal{N}(\overline{z}_o) = 0$, $\mathcal{N}(\frac{1}{z_o}) = 0$ and $\mathcal{N}(\frac{1}{\overline{z}_o}) = 0$.

From the previous remarks it follows that, for any $\varepsilon > 0$, there exists a constant $M' > 1$ such that, for all $M > M'$ and every $z \in \Delta$,

$$|z^{K-1} (\mathcal{N}(z) - \mathcal{N}_o(z))| < \varepsilon, \tag{18}$$

where Δ is any compact set of the open plane, $\mathcal{N}(z)$ is given by formula (16), while $\mathcal{N}_o(z)$ is defined as follows:

$$\mathcal{N}_o(z) = \lambda(K-1)K + \mu(N-1)N + \lambda \sum_{m=2}^{K} (K-m+1)^2 \left(z^{m-1} + \frac{1}{z^{m-1}}\right) + \tag{19}$$

$$+ \mu \sum_{m=2}^{N} (N-m+1)^2 \left(z^{m-1} + \frac{1}{z^{m-1}}\right).$$

We shall determine the zeros of the function $\mathcal{N}_o(z)$ on the circle $|z| = 1$. Since

$$\sum_{m=2}^{N} (N-m+1)^2 z^{-m+1} = \frac{1}{n^N} \sum_{m=2}^{N} (N-m+1)^2 z^{N-m+1} = \frac{1}{z^N} \sum_{n=1}^{N-1} n^2 z^n =$$

$$= \frac{1}{z^N} \left[((\sum_{n=1}^{N-1} z^n)' z)' z \right] = \frac{1}{z^N} \left[(\frac{z^N - z}{z-1})' z)' z \right] =$$

$$= \frac{1}{(z-1)^3} \left[(N-1)^2 z^2 - (2N^2-2N-1)z + N^2 - z^{-N+2} - z^{-N+1} \right],$$

therefore, proceeding analogously with the remaining addends of $\mathcal{N}_o(z)$, we get:

$$\mathcal{N}_0(z) = \frac{1}{(z-1)^3}\left\{\lambda\left[-K(z+1)^2(z-1) + z(z+1)(z^K - \frac{1}{z^K})\right] + \right.$$

$$\left. + \mu\left[-N(z+1)^2(z-1) + z(z+1)(z^N - \frac{1}{z^N})\right]\right\}.$$

Hence, after some transformations, we have:

$$\mathcal{N}_0(z) = \frac{(z+1)^2}{(z-1)^2} L_0(z), \tag{20}$$

where

$$L_0(z) = \lambda\left[\sum_{m=1}^{\frac{K-1}{2}} (z^{2m} + \frac{1}{z^{2m}}) - (K-1)\right] + \tag{21}$$

$$+ \mu\left[\sum_{m=1}^{\frac{N}{2}} (z^{2m-1} + \frac{1}{z^{2m-1}}) - N\right].$$

From (21) it can be seen at once that the only zero of the function $L_0(z)$ on the circle $|z| = 1$ is the point $z = 1$ which, in view of (19), is not a zero of $\mathcal{N}_0(z)$.

So, finally, it follows from (20) that the function $\mathcal{N}_0(z)$ has on the circle $|z| = 1$ one double zero $z = -1$ and $K - 2$ zeros inside as well as outside this circle.

Let us surround all zeros of the function $\mathcal{N}_0(z)$ with sufficiently small disjoint discs. From the Hurwitz theorem and condition (18) we infer that there exists some $M_0 > M'$ such that, for all $M > M_0$, zeros of the function $\mathcal{N}(z)$ given by formula (16) lie, respectively, in chosen neighbourhoods of zeros of the function $\mathcal{N}_0(z)$, with that in each of these neighbourhoods the number of zeros of both those functions, considering multiplicities, is the same.

It is well known [2] that the function $\mathcal{N}(z)$ has on the circle $|z| = 1$ at least one zero of even multiplicity. Let $\tilde{z} \neq 1$, $|\tilde{z}| = 1$, be one of these zeros. Then, for $M > M_0$, it lies in the vicinity of the double zero $z = -1$ of the function $\mathcal{N}_0(z)$. Since $\mathcal{N}(z)$ is a non-negative function of the circle

$|z| = 1$, the multiplicity of such a zero is at least 2; moreover, in the same neighbourhood there must lie a zero $\bar{\tilde{z}}$ of multiplicity at least 2, which contradicts the fact that the function $\mathcal{N}(z)$ must have exactly two zeros there, considering multiplicities. Consequently, $\tilde{z} = -1$ is the only zero of the function $\mathcal{N}(z)$ on the circle $|z| = 1$.

So, it results from the form of $\mathcal{N}(z)$ that, for $M > M_0$, this function can be represented as follows:

$$\mathcal{N}(z) = \frac{(z+1)^2}{z^{K-1}} L(z), \tag{22}$$

where $L(z)$ is some polynomial of degree $2K-4$, and $L(z) \neq 0$ for $|z| = 1$.

From the properties of the function $\mathcal{N}(z)$, given before, we know that, if $L(z_0) = 0$, then also $L(\bar{z}_0) = 0$, $L(\frac{1}{z_0}) = 0$ and $L(\frac{1}{\bar{z}_0}) = 0$.

We infer from equation (14) that the images $\tilde{w} = f(z)$ of zeros $\tilde{z}$, $|\tilde{z}| < 1$, of the function $\mathcal{N}(z)$ are zeros of the function $\mathcal{M}(w)$ since $f'(z) \neq 0$, whereas from the very form of the function $\mathcal{M}(w)$ it follows that also the points $\bar{\tilde{w}}$, $\frac{1}{\tilde{w}}$, $\frac{1}{\bar{\tilde{w}}}$ are its zeros. Besides, it is well known that the function $\mathcal{M}(w)$ has on the circle $|w| = 1$ at least one double zero w_0. From the above properties of the function $\mathcal{M}(w)$ we deduce that, for $M > M_0$,

$$\mathcal{M}(w) = \frac{(w - w_0)^2}{w^{K-1}} \hat{L}(w), \tag{23}$$

where $w_0 = -1$ or $w_0 = 1$, $\hat{L}(w)$ is some polynomial of degree $2K-4$, and $\hat{L}(w) \neq 0$ for $|w| = 1$.

To sum up, we have shown that, for $M > M_0$, every function $w = f(z) = \frac{1}{M} F(z)$, where F is an extremal function, satisfies equation (14), where $\mathcal{M}(w)$ and $\mathcal{N}(z)$ are given by formulae (23) and (22), respectively.

3.2. Determination of extremal function

From the R o y d e n theorem [8] one knows that every function $w = f(z) = \frac{1}{M} F(z)$ satisfying equation (14) maps the disc E onto the disc $|w| < 1$ lacking a finite number of analytic arcs $l_1, l_2, \ldots, l_j$, $j \geq 1$, with the following properties ([9], parts III, IV):

1° The arcs $l_1, l_2, \ldots, l_j$ lie in the disc $|w| < 1$ except, at most, their ends.

2° They are disjoint except, at most, their ends.

3° Each common point of the arc and the circle $|w| = 1$, or of two arcs, is a zero of the function $\mathcal{M}(w)$ given by formula (15); the number of arcs and their behaviour in the neighbourhood of such common point depend on the multiplicity of the zero (see [9], part III).

4° The union of the arcs $l_1, l_2, \ldots, l_j$ and of the circle $|w| = 1$ constitutes a continuum.

5° Along each of the arcs,

$$\text{(24)} \qquad \operatorname{Re} \int \sqrt{\mathcal{M}(w)}\, \frac{dw}{w} = \text{const},$$

where $\mathcal{M}(w)$ is a function defined by (15), and under the integral sign there occurs any branch of the root.

6° At least one of the ends of each arc is a zero of the function $\mathcal{M}(w)$ given by (15).

7° None of the arcs passes through the point $w = 0$.

We shall now prove that every function $w = f(z) = \frac{1}{M}F(z)$, where F is an extremal function in the class $S_R(M)$ for $M > M_0$, maps the disc E onto the dics $|w| < 1$ lacking one analytic arc with end at the point w_0. Really, let us take any function F extremal in $S_R(M)$ for $M > M_0$. Then the function $w = f(z) = \frac{1}{M} F(z)$ satisfies differential-functional equation (14), where the functions $\mathcal{M}(w)$ and $\mathcal{N}(z)$ are given by formulae (23) and (22), respectively, while the boundary of the image of the disc E under this mapping consists of the circle $|w| = 1$ and a finite number of analytic arcs described above.

Note that at least one of these arcs must have a common end with the circle $|w| = 1$, or else, the arcs along with the circle would not constitute a continuum. Without loss of generality, assume that l_1 is the arc. According to property 3^o, the common point of the arc l_1 and the circle $|w| = 1$ is a zero of the function $\mathcal{M}(w)$ given by (23). Since this function has on the circle $|w| = 1$ only one zero w_o, therefore l_1 must issue from the very point. The point w_o is a double zero of the function $\mathcal{M}(w)$, and it is well known ([9], p. 46) that at the double zero four arcs of (24), equally spaced at an angle of $\frac{\pi}{2}$, meet. Two of them are arcs of the circle $|w| = 1$, and consequently, of the remaining two, only one may enter the interior of the circle. This must be the arc l_1.

Note further that the union of the remaining arcs $l_2, \ldots, l_j$ is an empty set. For otherwise, the following cases would be possible: a) one of the arcs $l_2, \ldots, l_j$ has a common end $\tilde{w}_o \neq w_o$ with the circle $|w| = 1$, so, according to property 3^o, $\tilde{w}_o$ would have to be a zero of the function $\mathcal{M}(w)$ on the circle $|w| = 1$, which is impossible since the only zero of this function on $|w| = 1$ is the point w_o; b) any of the arcs $l_2, \ldots, l_j$ has a common end with the circle $|w| = 1$ at the point w_o, but then, at this point, more than four arcs of (24) would meet, which contradicts the fact that w_o is a double zero of the function $\mathcal{M}(w)$ on the circle $|w| = 1$; c) the end $\tilde{w}$ of the arc l_1, lying in the dics $|w| < 1$, is also an end of any of the arcs $l_2, \ldots, l_j$ and then, according to property 3^o, such point $\tilde{w}$ is a zero of function (23); but, as was noted earlier, each zero $\tilde{w}$ of the function $\mathcal{M}(w)$, lying in the disc $|w| < 1$, is the image of some zero $\tilde{z}$ of the function $\mathcal{N}(z)$, lying in the disc $|z| < 1$, so $\tilde{w}$ is an interior point of the image of the disc E under the mapping f, and consequently, it cannot lie on the boundary of this domain; d) none of the arcs $l_2, \ldots, l_j$ has common ends with the circle $|w| = 1$ and the arc l_1; this case is also impossible since, then, the union of the arcs $l_1, l_2, \ldots, l_j$ along with the circle $|w| = 1$ would not constitute a continuum, i.e., property 4^o would not hold.

Consequently, we have proved that the point w_o, $w_o = \pm 1$, is the end of the only cut l_1 in the image of the disc E under the mapping $w = f(z) = \frac{1}{M} F(z)$, where F is an extremal function in the class $S_R(M)$ for $M > M_o$.

It follows from the properties of the classes $S_R(M)$ considered that the image $f(E)$ of the disc E under the mapping $w = f(z) = \frac{1}{M}F(z)$ is symmetric with respect to the real axis, i.e., if $w \in f(E)$, then also $\overline{w} \in f(E)$.

Making use of the above fact, we shall show that the arc l_1 with end at the point $w_o = -1$ (or $w_o = 1$), symmetric with respect to the real axis, lies entirely on the real axis (cf. [4]). Without loss of generality, assume that $w_o = -1$.

Let $h(t)$ be a homeomorphism of the segment $\langle 0,1\rangle$ into the arc l_1, such that $h(0) = -1$. Suppose, despite of the announcement, that there exists a point $t_o \in (0,1\rangle$ such that $\operatorname{Im} h(t_o) \neq 0$, say, $\operatorname{Im} h(t_o) > 0$. Denote $T = \{t \in \langle 0,t_o): \operatorname{Im} h(t) = 0\}$. Of course,

$$t^* = \sup T \in T \quad \text{and} \quad t^* < t_o. \tag{25}$$

Besides, from the continuity of h:

$$\operatorname{Im} h(t) > 0 \quad \text{for} \quad t \in (t^*, t_o\rangle. \tag{26}$$

Since, for every point $h(t)$, $t \in \langle t^*, t_o\rangle$, the point $\overline{h(t)}$ belongs to the arc l_1, therefore there exists a continuous function

$$\hat{t} = \hat{t}(t) = h^{-1}(\overline{h(t)}), \quad t \in \langle t^*, t_o\rangle, \tag{27}$$

whose values range over an interval with endpoints

$$\hat{t}(t^*) = h^{-1}(\overline{h(t^*)}) = h^{-1}(h(t^*)) = t^*$$

and

$$\hat{t}(t_o) = h^{-1}(\overline{h(t_o)}) = \hat{t}_o. \tag{28}$$

From (26) and (27) it follows immediately that

$$\hat{t} \notin (t^*, t_o\rangle. \tag{29}$$

From this and (25):

$$\hat{t}_o < t^* \tag{30}$$

and, of course, $\operatorname{Im} h(\hat{t}_o) < 0$.

Let now $\hat{T} = \{t \in \langle 0, \hat{t}_o) : \operatorname{Im} h(t) = 0\}$. Of course,

$$\hat{t}^* = \sup \hat{T} \in \hat{T} \quad \text{and} \quad \hat{t}^* < \hat{t}_o. \tag{31}$$

From the continuity of h:

$$\operatorname{Im} h(t) < 0 \quad \text{for} \quad t \in (\hat{t}^*, \hat{t}_o\rangle. \tag{32}$$

Consider, as before, a continuous function

$$\hat{\hat{t}} = \hat{\hat{t}}(t) = h^{-1}\,(\overline{h(t)}), \quad t \in \langle \hat{t}^*, \hat{t}_o\rangle. \tag{33}$$

whose values now range over an interval with endpoints

$$\hat{\hat{t}}(\hat{t}^*) = h^{-1}(\overline{h(\hat{t}^*)}) = h^{-1}(h(\hat{t}^*)) = \hat{t}^*$$

and

$$\hat{\hat{t}}(\hat{t}_o) = h^{-1}(\overline{h(\hat{t}_o)}) = \hat{\hat{t}}_o.$$

From (32) and (33) it follows immediately that

$$\hat{\hat{t}} \notin (\hat{t}^*, \hat{t}_o\rangle.$$

From this and (31):

$$\hat{\hat{t}}_o < \hat{t}^*. \tag{34}$$

In view of (28),

$$\hat{\hat{t}}_o = h^{-1}(\overline{h(\hat{t}_o)}) = h^{-1}(h(t_o)) = t_o,$$

and consequently, taking account of the inequalities in (34), (31), (30) and (25), we obtain a contradiction.

To sum up, since the point $w = 0$ belongs to the image of the disc E under the mapping f, therefore, for $M > M_o$, every function $w = f(z) = \frac{1}{M} F(z)$, where F is an extremal function, maps the disc $|z| < 1$ onto the disc $|w| < 1$ lacking a segment on the real axis: a) with one end at the point $w_o = -1$ and the other one at some point of the negative real half-axis between -1 and 0, or b) with one end at the point $w_o = 1$ and the other one at some point of the positive real half-axis between 0 and 1. Consequently, from the properties of the Pick function $P_M(z)$ as well as from the Riemann theorem it follows that the only such function is in case: a) the function $p_M(z) = \frac{1}{M} P_M(z)$, whereas in case b) the function

$$-p_M(-z) = -\frac{1}{M} P_M(-z) = z + \sum_{n=2}^{\infty} (-1)^{n-1} P_{n,M} z^n,$$

where $P_M(z)$ is a Pick function. Since $P_{N,M} > 0$ for $M > M_o$, the inequality

$$\lambda P_{K,M} + \mu P_{N,M} > \lambda P_{K,M} - \mu P_{N,M}$$

is self-evident, and finally, the only extremal function realizing the maximum of functional (8) in the family $S_R(M)$ for $M > M_o$ is the Pick function $w = P_M(z)$ given by equation (3) and satisfying the condition $P_M(0) = 0$.

In the case when $N > K$, the proof of the theorem is analogous.

Consider in the family $S_R(M)$, $M > 1$, a real functional

$$\hat{J}(F) = \lambda_o A_{NF} + \sum_{j=1}^{m} \lambda_j A_{K_j F},$$

where m is any fixed positive integer, N - an even positive integer, K_j, $j = 1,2, \ldots, m$, - odd positive untegers, $\lambda_o > 0$, $\lambda_j \geqslant 0$, $j = 1,2, \ldots, m$.

From the theorem we have just proved follows

Corollary. There exists a constant $\hat{M}_o$, $\hat{M}_o > 1$, such that, for every $M > \hat{M}_o$ and every function $F \in S_R(M)$, the estimation

$$\hat{J}(F) \leqslant \lambda_o P_{N,M} + \sum_{j=1}^{m} \lambda_j P_{K_j,M}$$

holds, where

$$w = P_M(z) = z + \sum_{n=2}^{\infty} P_{n,M} z^n, \quad z \in E,$$

is a Pick function given by equation (3) and satisfying the condition $P_M(0) = 0$. It is the only function for which equality holds in the above estimation.

4. SUMMARY

The paper includes the following result:

Let $S_R(M)$, $M > 1$, be the class of functions

$$F(z) = z + \sum_{n=2}^{\infty} A_{nF} z^n$$

holomorphic and univalent in the disc $E = \{z : |z| < 1\}$, with real coefficients and such that, if $F \in S_R(M)$, then $|F(z)| < M$ for $z \in E$. Let further K, N be any fixed positive integers, K - odd, N - even; λ, μ - any real numbers, $\lambda > 0$, $\mu > 0$.

Then there exists a constant M_o, $M_o > 1$, such that, for all $M > M_o$ and every function $F \in \dot{S}_R(M)$, the estimation

$$\lambda A_{KF} + \mu A_{NF} \leqslant \lambda P_{K,M} + \mu P_{N,M} \tag{35}$$

is true, where

$$w = P_M(z) = z + \sum_{n=2}^{\infty} P_{n,M} z^n, \quad z \in E,$$

is a Pick function given by the equation

$$\frac{w}{\left(1 - \frac{w}{M}\right)^2} = \frac{z}{(1 - z)^2}, \quad z \in E,$$

and satisfying the condition $P_M(0) = 0$. This function is the only one for which, with a given M, $M > M_o$, equality holds in estimation (35).

From the theorem proved here follows the estimation $A_{NF} \leqslant$

$\leq P_{N,M}$, $N = 2,4,6, \ldots$, in the family $S_R(M)$, for M sufficiently large ([4], [5]).

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OGÓLNE OSZACOWANIE WSPÓŁCZYNNIKÓW FUNKCJI SYMETRYCZNYCH OGRANICZONYCH I JEDNOKROTNYCH

Praca zawiera następujący rezultat. Niech $S_R(M)$, $M > 1$, będzie klasą funkcji

$$F(z) = z + \sum_{n=2}^{\infty} A_{nF} z^n$$

holomorficznych, jednokrotnych w kole $E = \{z : |z| < 1\}$, o rzeczywistych współczynnikach i takich, że jeśli $F \in S_R(M)$, to $|F(z)| \leqslant M$ dla $z \in E$. Niech dalej K, N będą dowolnymi, ustalonymi liczbami naturalnymi, K - nieparzyste, N - parzyste; λ, μ - dowolnymi liczbami rzeczywistymi, $\lambda \geqslant 0$, $\mu > 0$. Wówczas istnieje stała M_o, $M_o > 1$, taka, że dla wszystkich $M > M_o$ i każdej funkcji $F \in S_R(M)$ prawdziwe jest oszacowanie

$$\lambda A_{KF} + \mu A_{NF} \leqslant \lambda P_{K,M} + \mu P_{N,M}, \tag{35}$$

gdzie

$$w = P_M(z) = z + \sum_{n=2}^{\infty} P_{n,M} z^n, \quad z \in E,$$

jest funkcją Picka daną równaniem

$$\frac{w}{\left(1 - \frac{w}{M}\right)^2} = \frac{z}{(1 - z)^2}, \quad z \in E,$$

i spełniającą warunek $P_M(0) = 0$. Funkcja ta jest jedyną, dla której przy danym M, $M > M_0$, zachodzi równość w oszacowaniu (35).