A C T A   U N I V E R S I T A T I S   L O D Z I E N S I S
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THE TURAN FUNCTIONAL FOR UNIVALENT POLYNOMIALS

In this note there has been obtained an effective exact estimation from below of some expression

$$T_p = \mathrm{Max}\left(\left|\frac{1}{w_1} + \frac{1}{w_2}\right|, \left|\frac{1}{w_1^2} + \frac{1}{w_2^2}\right|\right),$$

for the parameters w_1, w_2 defining the canonical univalent polynomials

$$P(w) = \int_0^w \left(1 - \frac{w}{w_1}\right)\left(1 - \frac{w}{w_2}\right) dw.$$

INTRODUCTION

In various branches of analysis, the estimation from below of expressions of type

$$\underset{\nu = 1, \ldots, n}{\mathrm{Max}} | z_1^{\nu} + \ldots + z_n^{\nu} |$$

for some sets of complex arguments $z_1, \ldots, z_n$ finds its application. They were dealt with by, in particular, T u r a n, C a s s e l s, A t k i n s o n, Ł a w r y n o w i c z and others, with that it is here the question of both universal estimations with every n, not necessarily sharp (cf. [1], [2], [9], [10]), and estimations for specified n, exact (cf. [7], [8]).

In the present note we give an exact estimation of this expression for $n = 3$, for arguments corresponding to canonical univalent polynomials (cf. [3] and [5]).

I. TERMS, NOTATIONS AND AUXILIARY NOTES

1. Polynomials of the form

$$P(w) = w + c_2 w^2 + \ldots + c_{m+1} w^{m+1}, \quad m \geqslant 1, \tag{1.1}$$

will be called normalized. From the definition it can be seen that such polynomials do not reduce to constants, and that zeros their derivatives are different from 0.

2. Let w_k, $k = 1, \ldots, n$, stand for distinct zeros of the derivative of a polynomial of type (1.1), whereas γ_k, $k = 1, \ldots, n$, their multiplicities. In that case, polynomial (1.1) can be represented in the form

$$P(w) = \int_0^w \prod_{k=1}^{n} \left(1 - \frac{w}{w_k}\right)^{\gamma_k} dw. \tag{1.2}$$

With the above notations, let still

$$\xi_k = P(w_k), \quad k = 1, \ldots, n. \tag{1.3}$$

Then, between the position of the zeros w_k and of the values ξ_k, the relations

$$\frac{1}{4} \min_{1 \leqslant k \leqslant n} |\xi_k| \leqslant |w_k| \leqslant (m+1)4^{m+1} \max_{1 \leqslant k \leqslant n} |\xi_k|, \tag{1.4}$$
$$k = 1, \ldots, n,$$

hold (cf. [5] part II, th. 1).

3. For given polynomials (1.1), let μ_P denote the greatest domain of the complex plane, containing the point 0 and such that $|P(w)| < 1$ for $w \in \mu_P$. The domain μ_P will be called a domain associated with polynomial (1.1). Directly from the definition of the domain μ_P it follows that $|P(w)| = 1$ for $w \in \partial u_P$.

4. In order that polynomial (1.1) be univalent, it is necessary and sufficient that all zeros of its derivative lie outside μ_P (cf. [3], lemma 4).

5. Polynomial (1.1) will be called canonical when all zeros of its derivative lie on the boundary of the associated domain μ_P, that is, when

(1.5) $$w_k \in \partial u_P, \quad \text{i.e.} \quad |P(w_k)| = 1, \quad k = 1, \ldots, n.$$

Such polynomials do exist, in particular, such are those which realize the extrema of some functionals (cf. [3], th. 5). We notice that, in accordance with (1.2), condition (1.5) can be written in the form

(1.5′) $$\left| \int_0^{w_k} \prod_{k=1}^{n} \left(1 - \frac{w}{w_k}\right)^{\gamma_k} dw \right| = 1,$$

and relations (1.4), on account of (1.3) and (1.5), in the form

(1.4′) $$\frac{1}{4} \leqslant |w_k| \leqslant (m+1)4^{m+1}, \quad k = 1, \ldots, n.$$

6. The family of all normalized univalent polynomials of type (1.1) considered in associated domains, of degree not greater than M where $M \geqslant 1$, will be denoted by γ_M (cf. [3]). $\hat{\gamma}_M$ will stand for the family of all canonical polynomials of degree M belonging to γ_M. The families γ_M and $\hat{\gamma}_M$ are compact (cf. [3], p. 16 and [4], lemma 8 and [6]).

7. If $\{a_s\}$, $\{b_s\}$ are sequences with non-negative terms, converging to a and b, respectively, then, as can easily be seen,

(1.6) $$\lim_{s \to \infty} \sqrt[s]{a_s^s + b_s^s} = \text{Max}\,(a,b).$$

Hence, if $a_s^s + b_s^s \neq 0$, $s = 1, 2, \ldots$ and at least one of the numbers a,b is different from zero, then

(1.6′) $$\lim_{s\to\infty} \frac{1}{s}\log\,(a_s^s + b_s^s) = \log \operatorname{Max}\,(a,b).$$

II. ESTIMATION OF THE MAXIMUM FROM BELOW

Let us consider on the family $\hat{\gamma}_3$ a functional

(2.1) $$T_P = \operatorname{Max}\left(\left|\frac{1}{w_1} + \frac{1}{w_2}\right|,\ \left|\frac{1}{w_1^2} + \frac{1}{w_2^2}\right|\right),$$

where $P \in \hat{\gamma}_3$, and w_1, w_2 are zeros of the derivative of P. As can be seen, this functional is continuous and different from zero.

Theorem. For each polynomial $P \in \hat{\gamma}_3$, the sharp estimation

(2.2) $$T_P \geqslant \frac{2\sqrt{3}}{27}\sqrt{(7-a)(5+a)}$$

holds, where $a = \sqrt[3]{197 + \sqrt{41\ 553}} + \sqrt[3]{197 - \sqrt{41\ 553}}$.

The extremal polynomials realizing the equality sign are of the form

(2.3) $$P^*(w) = w - \frac{\sqrt{3(5+a)}}{81}\left(7 - a + \varepsilon\sqrt{(7-a)(2+a)}\, i\right) e^{-i\theta} w^2 + \frac{5+a}{729}\left(5 - 2a + 2\varepsilon\sqrt{(7-a)(2+a)}\, i\right) e^{-2i\theta} w^3,$$

where $\varepsilon = \pm 1$, θ - a real number.

P r o o f. We shall carry it out by three stages given below.

1. Auxiliary and limit polynomials and parameters

Having regard to 7, let us consider on $\hat{\gamma}_3$, for any momently fixed s, an auxiliary functional

(2.4) $$F_P^s = \frac{1}{2s} \log \left(\left| \frac{1}{w_1} + \frac{1}{w_2} \right|^{2s} + \left| \frac{1}{w_1^2} + \frac{1}{w_2^2} \right|^{2s} \right), \quad P \in \hat{\gamma}_3,$$

w_1, w_2 are zeros of the derivative of P. This functional is, as can be seen, defined and continuous; since the family $\hat{\gamma}_3$ is compact, there exists an auxiliary polynomial $P_s^* \in \hat{\gamma}_3$ which realizes Minimum of (2.4), that is, for each $P \in \hat{\gamma}_3$, the inequality

(2.5) $$F_P^s \geqslant F_{P_s^*}$$

is true, i.e.

(2.5') $$\frac{1}{2s} \log \left(\left| \frac{1}{w_1} + \frac{1}{w_2} \right|^{2s} + \left| \frac{1}{w_1^2} + \frac{1}{w_2^2} \right|^{2s} \right) \geqslant$$
$$\frac{1}{2s} \log \left(\left| \frac{1}{w_{1s}^*} + \frac{1}{w_{2s}^*} \right| + \left| \frac{1}{w_{1s}^{*2}} + \frac{1}{w_{2s}^{*2}} \right| \right),$$

where the auxiliary parameters w_1, w_2 are zeros of the derivative of P, and w_{1s}^*, w_{2s}^* - zeros of the derivative of P_s^*. In view of the compactness of the family $\hat{\gamma}_3$ and (1.4'), choosing subsequences if necessary, one may assume that the parameters and polynomials

(2.6) $$\frac{1}{w_{1s}^*}, \quad \frac{1}{w_{2s}^*}, \quad P_s^*, \quad s = 1, 2, \ldots,$$

converge to some limit ones

(2.7) $$\frac{1}{w_1^*}, \quad \frac{1}{w_2^*}, \quad P^*,$$

of which the first two are finite and different from zero according to (1.4'), whereas the third is a canonical polynomial of the family $\hat{\gamma}_3$, with zeros w_1^*, w_2^* of its derivative. Passing to the limit in (2.5′), we have, according to (1.6′), for each $P \in \hat{\gamma}_3$,

(2.8)
$$\log \operatorname{Max}\left(\left|\frac{1}{w_1}+\frac{1}{w_2}\right|, \left|\frac{1}{w_1^2}+\frac{1}{w_2^2}\right|\right) \geq \log \operatorname{Max}\left(\left|\frac{1}{w_1^*}+\frac{1}{w_2^*}\right|, \left|\frac{1}{w_1^{*2}}+\frac{1}{w_2^{*2}}\right|\right),$$

i.e.

(2.8′)
$$\operatorname{Max}\left(\left|\frac{1}{w_1}+\frac{1}{w_2}\right|, \left|\frac{1}{w_1^2}+\frac{1}{w_2^2}\right|\right) \geq \operatorname{Max}\left(\left|\frac{1}{w_1^*}+\frac{1}{w_2^*}\right|, \left|\frac{1}{w_1^{*2}}, \frac{1}{w_2^{*2}}\right|\right),$$

that is, coming back to (2.1), we have $T_P \geq T_{P^*}$. The polynomial P^* realizes, of course, the equality signs in (2.8) and (2.8′), i.e. minimum of functional (2.1).

Consequently, the justification of (2.2) is reduced to the investigation and determination of auxiliary and limit polynomials and parameters as well as values of (2.1) corresponding to them.

2. Equations for auxiliary parameters

We now notice that if w_1, w_2 are arbitrary and sufficiently close to w_{1s}^*, w_{2s}^* and satisfy the conditions

(2.9)
$$\left|\int_0^{w_j} \left(1-\frac{w}{w_1}\right)\left(1-\frac{w}{w_2}\right) dw\right| = 1, \quad j = 1, 2,$$

then inequality (2.4′) holds. Indeed, in the case under consideration the polynomial

$$P(w) = \int_0^{w} \left(1-\frac{w}{w_1}\right)\left(1-\frac{w}{w_2}\right) dw$$

is sufficiently close to the canonical one

$$P_s^*(w) = \int_0^w (1 - \frac{w}{w_{1s}^*})(1 - \frac{w}{w_{2s}^*})\, dw$$

and satisfies conditions (2.9), i.e. it is canonical (cf. [5], part I, th. 1). In consequence, it satisfies inequality (2.5), i.e. (2.5').

In this situation, by putting

(2.10) $$f_s(w_1, w_2, \overline{w}_1, \overline{w}_2) =$$

$$= \frac{1}{2s} \log\left(\left(\frac{1}{w_1} + \frac{1}{w_2}\right)^s \left(\frac{1}{\overline{w}_1^2} + \frac{1}{\overline{w}_2^2}\right)^s + \right.$$

$$\left. + \left(\frac{1}{w_1^2} + \frac{1}{w_2^2}\right)^s \left(\frac{1}{\overline{w}_1^2} + \frac{1}{\overline{w}_2^2}\right)^s\right)$$

and

(2.11) $$g_1(w_1, w_2, \overline{w}_1, \overline{w}_2) =$$

$$= \left(w_1 - \frac{1}{3}\frac{w_1^2}{w_2}\right)\left(\overline{w}_1 - \frac{1}{3}\frac{\overline{w}_1^2}{\overline{w}_2}\right) - 4,$$

(2.12) $$g_2(w_1, w_2, \overline{w}_1, \overline{w}_2) =$$

$$= \left(w_2 - \frac{1}{3}\frac{w_2^2}{w_1}\right)\left(\overline{w}_2 - \frac{1}{3}\frac{\overline{w}_2^2}{\overline{w}_1}\right) - 4,$$

from the above we get at once that function (2.10) attains at the point $(w_{1s}^*, w_{2s}^*, \overline{w}_{1s}^*, \overline{w}_{2s}^*)$ the local minimum associated, under the conditions for the vanishing of functions (2.11) and (2.12), in the sense as described in [4], p. 6 and 7. This enables one to apply the known result (cf. [4], lemma 4) according to which, there exists a non-trivial system of real multipliers τ'_s, λ'_{1s}, λ'_{2s}, corresponding to conditions (2.9), such that at the extremal point $(w^*_{1s}, w^*_{2s}, \overline{w}^*_{1s}, \overline{w}^*_{2s})$, the equations

$$\tau'_s \frac{\partial f_s}{\partial w_j} = \lambda'_{1s} \frac{\partial g_1}{\partial w_1} + \lambda'_{2s} \frac{\partial g_2}{\partial w_j}, \quad j = 1, 2,$$

as well as (2.9) are satisfied; hence, in conformity with notations (2.10), (2.11), (2.12) and (2.9), after carrying out some simple calculations and introducting notations $\tau_s = -\frac{1}{2}\tau'_s$, $\lambda_{1s} = 4\lambda'_{1s}$, $\lambda_{2s} = 4\lambda'_{2s}$, we obtain the equations

$$\tau_s \frac{\frac{1}{w^*_{js}}\left|\frac{1}{w^*_{1s}} + \frac{1}{w^*_{2s}}\right|^{2s-2}\left(\frac{1}{\overline{w}^*_{1s}} + \frac{1}{\overline{w}^*_{2s}}\right) + \frac{2}{w^{*2}_{js}}\left|\frac{1}{w^{*2}_{1s}} + \frac{1}{w^{*2}_{2s}}\right|^{2s-2}\left(\frac{1}{\overline{w}^{*2}_{1s}} + \frac{1}{\overline{w}^{*2}_{2s}}\right)}{\left|\frac{1}{w^*_{1s}} + \frac{1}{w^*_{2s}}\right|^{2s} + \left|\frac{1}{w^{*2}_{1s}} + \frac{1}{w^{*2}_{2s}}\right|^{2s}} =$$

(2.13)

$$\lambda_{js} \frac{1 - \frac{2}{3}\frac{w^*_{js}}{w^*_{ks}}}{1 - \frac{1}{3}\frac{w^*_{js}}{w^*_{ks}}} + \lambda_{ks} \frac{\frac{1}{3}\frac{w^*_{ks}}{w^*_{js}}}{1 - \frac{1}{3}\frac{w^*_{ks}}{w^*_{js}}}$$

and

(2.14) $$\left|w^*_{js} - \frac{1}{3}\frac{w^{*2}_{js}}{w^*_{ks}}\right|^2 = 4, \quad j = 1, 2, \quad k = 3 - j.$$

Finally, by applying the evident normalization, it can be assumed that

(2.15) $$\tau_s^2 + \lambda_{1s}^2 + \lambda_{2s}^2 = 1, \quad s = 1, 2, \ldots$$

(2.13)-(2.15) represent the sought - for equations for auxiliary parameters.

3. Equations for limit parameters

Let us now put

$$(2.16)\qquad M_s = \mathrm{Max}\left(\left|\frac{1}{w^*_{1s}} + \frac{1}{w^*_{2s}}\right|,\ \left|\frac{1}{w^{*2}_{1s}} + \frac{1}{w^{*2}_{2s}}\right|\right);$$

then, in accordance with (2.7), there exists a limit

$$(2.17)\qquad \lim_{s\to\infty} M_s = M^* = \mathrm{Max}\left(\left|\frac{1}{w^*_1} + \frac{1}{w^*_2}\right|,\ \left|\frac{1}{w^{*2}_1} + \frac{1}{w^{*2}_2}\right|\right)$$

different from zero; at the same time, after choosing (if necessary) convergent subsequences, one may assume that there exist limits

$$(2.18)\qquad \tau = \lim_{s\to\infty} \tau_s,\quad \lambda_j = \lim_{s\to\infty} \lambda_{js},\quad j = 1,\ 2,$$

and

$$(2.19)\qquad \lim_{s\to\infty} \left|\frac{1}{w^*_{1s}} + \frac{1}{w^*_{2s}}\right|^{2s-2} : M_s^{2s-2} = \varrho_1,$$

$$\lim_{s\to\infty} \left|\frac{1}{w^{*2}_{1s}} + \frac{1}{w^{*2}_{2s}}\right|^{2s-2} : M_s^{2s-2} = \varrho_2,$$

of which (2.18) are finite and not all zero by (2.15), whereas (2.19), according to (2.16), are not all zero, either, and satisfy the conditions

$$(2.20)\qquad 0 \leqslant \varrho_j \leqslant 1,\quad j = 1,\ 2.$$

What is more, we obtain notice that if, for some j, there is

$$(2.21)\qquad \varrho_j \neq 0,\ \text{ then } \left|\frac{1}{w^{*j}_1} + \frac{1}{w^{*j}_2}\right| : M^* = 1.$$

From this and the above it follows that there is always

$$\varrho_1 \left| \frac{1}{w_1^*} + \frac{1}{w_2^*} \right| + \varrho_2 \left| \frac{1}{w_1^{*2}} + \frac{1}{w_2^{*2}} \right| \neq 0.$$

In the light of what has been said above, we may now pass to the limit in (2.13)-(2.15). Taking account of notations (2.18) and (2.19), we get the equations

$$\tau \frac{\frac{1}{w_j^*}\left(\frac{1}{\overline{w}_1^*} + \frac{1}{\overline{w}_2^*}\right)\varrho_1 + \frac{2}{w_j^{*2}}\left(\frac{1}{\overline{w}_1^{*2}} + \frac{1}{\overline{w}_2^{*2}}\right)\varrho_2}{\left|\frac{1}{w_1^*} + \frac{1}{w_2^*}\right|^2 \varrho_1 + \left|\frac{1}{w_1^{*2}} + \frac{1}{w_2^{*2}}\right|^2 \varrho_2} =$$

(2.22)

$$\lambda_j \frac{1 - \frac{2}{3}\frac{w_j^*}{w_k^*}}{1 - \frac{1}{3}\frac{w_j}{w_k}} + \lambda_k \frac{\frac{1}{3}\frac{w_k^*}{w_j^*}}{1 - \frac{1}{3}\frac{w_k}{w_j}} .$$

and

(2.23) $$|w_j^*|^2 \left|1 - \frac{1}{3}\frac{w_j^*}{w_k^*}\right|^2 = .4, \quad j = 1, 2, \quad k = 3 - j,$$

(2.24) $$\tau^2 + \lambda_1^2 + \lambda_2^2 = 1.$$

In order to examine these equations, we put

(2.25) $$t = w_1^* / w_2^* .$$

We notice that, according to (2.7), ratio (2.25) is finite and unequal to zero. With notation (2.25), equations (2.23) take the form

(2.26) $$|w_1^*|^2 \; |1 - \tfrac{1}{3} t|^2 = 4,$$

(2.27) $$|w_2^*|^2 \; |1 - \tfrac{1}{3t}|^2 = 4.$$

From the equations obtained it is first seen that $t \neq 3$ and $t \neq$ $\neq 1/3$ and next, by dividing them, we get:

$$t\bar{t} = 1 \text{ or} \tag{2.28'}$$

$$t^2\bar{t}^2 - 3(t\bar{t} + 1)(t + \bar{t}) + 10t\bar{t} + 1 = 0. \tag{2.28"}$$

The examination of equations (2.22) will be carried out in the cases given below.

a. $\tau = 0$. Then equations (2.22), after taking (2.25) into account, take the form

$$\lambda_1 \frac{3 - 2t}{3 - t} + \lambda_2 \frac{1}{3t - 1} = 0, \tag{2.29}$$

$$\lambda_1 \frac{t}{3 - t} + \lambda_2 \frac{3t - 2}{3t - 1} = 0. \tag{2.30}$$

After adding them up, we get

$$\lambda_1 + \lambda_2 = 0.$$

From this and (2.24) it follows that $\lambda_1 \neq 0$ and $\lambda_2 \neq 0$. The elimination of these multipliers from (2.29) and (2.30) gives

$$-6t^2 + 12t - 6 = 0,$$

that is,

$$t = 1.$$

Hence and from (2.26) and (2.27) we obtain

$$w_1^* = 3e^{i\theta}, \quad w_2^* = 3e^{i\theta}, \quad \theta \text{ - a real number.}$$

Substituting the found values w_1^*, w_2^* into (2.8'), we get

$$\operatorname{Max}\left(\left|\frac{1}{w_1^*} + \frac{1}{w_2^*}\right|, \left|\frac{1}{w_1^{*2}} + \frac{1}{w_2^{*2}}\right|\right) = \frac{2}{3}. \tag{2.31}$$

b. $\tau \neq 0$. Then, applying division (if necessary), one may adopt $\tau = 1$. On account of conditions (2.20), we distinguish three subcases.

(i) $\varrho_1 \neq 0$, $\varrho_2 = 0$. Then, in virtue of (2.21), we get

$$t \neq -1. \tag{2.32}$$

In view of that, equations 2.22 take the form

$$\frac{1}{t+1} = \lambda_1 \frac{3-2t}{3-t} + \lambda_2 \frac{1}{3t-1}, \tag{2.33}$$

$$\frac{t}{t+1} = \lambda_1 \frac{3}{3-t} + \lambda_2 \frac{3t-2}{3t-1}. \tag{2.34}$$

Adding them up, we obtain

$$\lambda_1 + \lambda_2 = 1. \tag{2.35}$$

From (2.35) it follows that λ_1 and λ_2 are not all zero. The equations: (2.33), the one conjugate to it and (2.35) can be treated as linear equations homogeneous with respect to λ_1, λ_2, 1. The elimination of the multipliers λ_1, λ_2 and 1 from then gives, after comparing the corresponding determinant to zero,

(2.36′) $t = 1$, or (2.36″) $t = \bar{t}$, or (2.36‴) $t\bar{t} +$

$$- 3(t + \bar{t}) + 1 = 0$$

Now, it is necessary to consider six systems of equations, resulting from (2.28′) or (2.28″) and (2.36′) or (2.36″) or (2.36‴). Cases (2.28′) and (2.36′), (2.28′) and (2.36″), (2.28″) and (2.36′) need not be worked out.

In case (2.28″) and (2.36″), by eliminating $\bar{t}$, we obtain

$$t^4 - 6t^3 + 10t^2 - 6t + 1 = 0,$$

next, in case (2.28′) and (2.36‴), by eliminating $\bar{t}$, we get

$$3\left(t + \frac{1}{t}\right) - 2 = 0,$$

and finally, in case (2.28") and (2.36"′), by eliminating $t+\bar{t}$, we obtain an inconsistent equation.

The solving of the equations in the five cases in question gives, respectively,

$$t = 1; \quad t = 1 \text{ or } t = -1; \quad t = 1;$$

$$t = 1 \text{ or } t = 1 \text{ or } t = 2 + \varepsilon\sqrt{3},$$

$$\varepsilon = \pm 1; \quad t = \frac{1}{3} + \varepsilon\frac{2}{3}\sqrt{2}i, \quad \varepsilon = \pm 1.$$

According to (2.32), the relation $t = -1$ is impossible. In the remaining cases, on the ground of equation (2.26), we have five contingencies

$$w_1^* = 3e^{i\theta}, \quad w_2^* = 3e^{i\theta}; \quad w_1^* = 3(\sqrt{3}+\varepsilon)e^{i\theta}, \quad w_2^* = 3(\sqrt{3}-\varepsilon)e^{i\theta};$$

$$w_1^* = \frac{3\sqrt{2}}{2}e^{i\theta}, \quad w_2^* = \frac{\sqrt{2}}{2}(1 - \varepsilon 2\sqrt{2}\, i)e^{i\theta},$$

where θ - a real number. For the found pairs w_1^*, w_2^*, the expression on the right-hand side of (2.8′) takes the values

(2.37) $$\frac{2}{3}, \quad \frac{\sqrt{3}}{3}, \quad \frac{4}{9}\sqrt{3},$$

respectively.

(ii) $\varrho_1 = 0$, $\varrho_2 = 1$. Then, by (2.21), we get

(2.38) $$t^2 \neq -1.$$

In view of this fact, equations (2.22) take the form

$$\frac{1}{t^2+1} = \lambda_1\frac{3-2t}{3-t} + \lambda_2\frac{1}{3t-1},$$

$$\frac{t^2}{t^2+1} = \lambda_1\frac{t}{3-t} + \lambda_2\frac{3t-2}{3t-1}.$$

Proceeding quite analogously as before, we obtain

(2.39′) $t = 1$, or (2.39″) $t = \bar{t}$, or (2.39‴) $t^2\bar{t}^2 -$

$$- 5(t\bar{t} + 1)(t + \bar{t}) + 6(t + \bar{t})^2 - 2t\bar{t} + 1 = 0.$$

Similarly as before, one ought to consider six systems of equations, resulting from (2.28′) or (2.28″) and (2.39′) or (2.39″) or (2.39‴). The first four cases are identical as previously; in case (2.28′) and (2.29‴), by eliminating $\bar{t}$, we obtain

$$\left(t + \frac{1}{t}\right)\left(3\left(t + \frac{1}{t}\right) - 5\right) = 0,$$

next, in case (2.28″) and (2.39‴), by eliminating $t + \bar{t}$, we get

$$5t^2\bar{t}^2 - 22t\bar{t} + 5 = 0.$$

The solving of the equations in the discussed six cases gives

$$t = 1; \quad t = 1 \quad \text{or} \quad t = -1; \quad t = 1; \quad t = 1; \quad \text{or} \quad t = 1$$

$$\text{or } t = 2 + \varepsilon\sqrt{3}, \quad \varepsilon = \pm 1; \quad t = -i \quad \text{or} \quad t = i \quad \text{or} \quad t = \frac{5 + \varepsilon\sqrt{11}\, i}{6},$$

$$\varepsilon = \pm 1; \quad t = \frac{4 + \varepsilon\sqrt{6}}{10}(3 + \delta_1), \quad \varepsilon = \pm 1, \quad \delta = \pm 1.$$

In accordance with (2.38), the relations $t = -i$ and $t = i$ are impossible.

In the remaining cases, on the ground of equation (2.26), we have ten contingencies

$$w_1^* = 3e^{i\theta}, \quad w_2^* = 3e^{i\theta}; \quad w_1^* = \frac{3}{2}e^{i\theta}, \quad w_2^* = -\frac{3}{2}e^{i\theta}; \quad w_1^* = 3(\sqrt{3} + \varepsilon)e^{i\theta},$$

$$w_2^* = 3(\sqrt{3} - \varepsilon)e^{i\theta}; \quad w_1^* = \frac{6\sqrt{5}}{5}e^{i\theta}, \quad w_2^* = \frac{\sqrt{5}}{5}(5 - \varepsilon\sqrt{11}\, i)e^{i\theta};$$

$$w_1^* = 6\sqrt{\frac{4 + \varepsilon\sqrt{6}}{10}}\, e^{i\theta}, \quad w_2^* = \frac{3}{5}\sqrt{4 - \varepsilon\sqrt{6}}\,(3 - \delta_1)e^{i\theta},$$

where θ is a real number. For the found pairs w_1^*, w_2^*, the expressions $\left|\frac{1}{w_1^*} + \frac{1}{w_2^*}\right|$ and $\left|\frac{1}{w_1^{*2}} + \frac{1}{w_2^{*2}}\right|$ take the values

$$\frac{2}{3},\ \frac{4}{3},\ \frac{\sqrt{3}}{3},\ \frac{\sqrt{5}\,\sqrt{33}}{18},\ \frac{\sqrt{14}}{16}\quad\text{and}\quad \frac{2}{9},\ \frac{8}{9},\ \frac{2}{9},\ \frac{25}{108},\ \frac{\sqrt{15}}{18},$$

respectively, which indicates that

$$M^* = \left|\frac{1}{w_1^*} + \frac{1}{w_2^*}\right| > \left|\frac{1}{w_1^{*2}} + \frac{1}{w_2^{*2}}\right|,$$

whereas in the case under consideration, in virtue of (2.21), there, should be

$$\left|\frac{1}{w_1^{*2}} + \frac{1}{w_2^{*2}}\right| : M^* = 1.$$

(iii) $\varrho_1 \neq 0$, $\varrho_2 \neq 0$. Then, according to (2.21), we have

$$\left|\frac{1}{w_1^*} + \frac{1}{w_2^*}\right| : M^* = 1 \quad\text{and}\quad \left|\frac{1}{w_1^{*2}} + \frac{1}{w_2^{*2}}\right| : M^* = \ = 1,$$

that is,

$$(2.40)\qquad \frac{1}{|w_1^*|}|1 + t| = \frac{1}{|w_2^*|^2}|J + t^2|.$$

After eliminating $|w_1^*|$ from equations (2.26) and (2.40), we get

$$(2.41)\qquad (t^2\bar{t}^2 + (t + \bar{t})^2 - 2t\bar{t} + 1)(\tfrac{1}{9}\,t\bar{t} - \tfrac{1}{3}(t + \bar{t}) + 1) -$$

$$- 4\,(t\bar{t} + t + \bar{t} + 1) = 0.$$

It is now necessary to consider two systems of equations (2.28′) and (2.41), and (2.28″) and (2.41). We shall first deal with the latter. The elimination of $t + \bar{t}$ gives, for $t\bar{t}$, the equation

$$11(t\bar{t})^4 + 76(t\bar{t})^3 + 114(t\bar{t})^2 + 76(t\bar{t}) + 11 = 0$$

which, as is easily seen, possesses no positive roots. System (2.28′) and (2.41) gives the equation

$$(t + \bar{t})^3 - \frac{10}{3}(t + \bar{t})^2 + 12(t + \bar{t}) + 24 = 0$$

which, as can easily be verified, possesses on real root. Applying the Cardano formulae, we get

$$t + \bar{t} = \frac{10}{9} - \frac{4}{9}\left(\sqrt[3]{197 - \sqrt{41\ 553}} + \sqrt[3]{197 + \sqrt{41\ 553}}\right).$$

From this and (2.28′) we obtain

$$t = \frac{5}{9} - \frac{2}{9}a + \varepsilon\frac{2}{9}\sqrt{(7 - a)(2 + a)}\, i', \quad \varepsilon = \pm 1,$$

where

$$a = \sqrt[3]{197 - \sqrt{41\ 553}} + \sqrt[3]{197 + \sqrt{41\ 553}}\ .$$

The found t and equation (2.26) give

$$w_1^* = \frac{9}{\sqrt{3(5 + a)}}\, e^{i\theta}, \tag{2.42}$$

$$w_2^* = \frac{5 - 2a - 2\varepsilon\sqrt{(7 - a)(2 + a)}\, i}{\sqrt{3(5 + a)}}\, e^{i\theta}, \quad \theta - \text{real}.$$

For the pair w_1^*, w_2^* obtained, the expression on the right-hand side of (2.1) takes the value

$$\frac{2\sqrt{3}}{27}\sqrt{(7 - a)(5 + a)}. \tag{2.43}$$

To sum up, from (2.31), (2.37) and (2.43) we have that the expression on the right-hand side of (2.8′), and thus of (2.1), is no less than

$$\operatorname{Min}\left(\frac{2}{3}, \frac{\sqrt{3}}{3}, \frac{4}{9}\sqrt{3}, \frac{2\sqrt{3}}{27}\sqrt{(7 - a)(5 + a)}\right).$$

It should still be shown that inequality (2.2) is sharp. For the purpose, it suffices to put values (2.42) in formula (1.2) to obtain all polynomials realizing in (2.2) the equality sign. After simple calculations we get that they are of the form (2.3).

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FUNKCJONAŁ TURANA DLA WIELOMIANÓW JEDNOKROTNYCH

Uzyskuje się efektywne, dokładne oszacowanie od dołu wyrażenia

(1)
$$T_p = \mathrm{Max}\left(\left|\frac{1}{w_1} + \frac{1}{w_2}\right|, \quad \left|\frac{1}{w_1^2} + \frac{1}{w_2^2}\right|\right),$$

dla parametrów w_1, w_2 określających wielomiany jednokrotne kanoniczne

$$P(w) = \int_0^w \left(1 - \frac{w}{w_1}\right)\left(1 - \frac{w}{w_2}\right) dw.$$

Ograniczenie wynosi

$$\frac{2\sqrt{3}}{27}\sqrt{(7 - a)(5 + a)},$$

gdzie $a = \sqrt[3]{197 + \sqrt{41\,553}} + \sqrt[3]{197 - \sqrt{41\,553}}$.

Parametry ekstremalne realizujące znak równości wynoszą

$$w_1^* = \frac{9}{\sqrt{3(5 + a)}} e^{i\theta}, \quad w_2^* = \frac{5 - 2a - 2\varepsilon\sqrt{(7 - a)(2 + a)}\, i}{\sqrt{3}\,(5 + a)} e^{i\theta},$$

gdzie θ - liczba rzeczywista, $\varepsilon = \pm 1$.

Dowód oparty jest na własnościach wielomianów kanonicznych (por. [5] i [6]) i wyznaczaniu ekstremum lokalnego przy warunkach pobocznych zespolonych (por. [4] lemat 4).

Wyrażenia typu (1) dla parametrów z koła jednostkowego były badane ogólnie między innymi przez T u r a n a i efektywnie przez Ł a w r y n o w i c z a (por. [7], [8], [9]).