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SOME PROPERTIES OF SUBSETS OF R^k WITH THE BAIRE PROPERTY

In the paper there has been obtained a generalization of some theorem of Kuczma on the form of a sequence $\{Z_n\}_{n \in N}$ of sets constituting a decomposition of the space R^k, under the assumption that each of them possesses the Baire property and satisfies the conditions $Z_n + Z_n = Z_n$ and $Z_n \cap Z_m = \{0\}$ for $n \neq m$.

Here is considered a countable family $\{Z_n\}_{n=1}^{\infty}$ of subsets of R^k fulfiling the conditions

$$Z_n + Z_n = Z_n, \quad 0 \in Z_n,$$

where $Z_n + Z_n$ denotes the set of all sums $z_1 + z_2$ with $z_1 \in Z_n$, $z_2 \in Z_n$. Kuczma proved in [1] the following theorem.

Theorem 1. If a set $Z \subset R^k$, $Z \neq R^k$, fulfils $Z + Z = Z$, $0 \in Z$, $Z + Z = Z$ and has the Baire property, then there exists an $(k - 1)$ dimensional hyperplane H passing through the origin, such that $Z = P \cup Z_0$ where P is one of two open half-spaces determined by H and Z_0 is a quite arbitrary subset of H fulfilling the condition $Z_0 + Z_0 = Z_0$.

We shall extend this theorem by replacing the two sets Z, $R^k - Z$ with a countable family of subsets of R^k. The following lemma may by obtained analogically to *Lemma 1* in [1].

Lemma 1. If $Z \subset R$ is a set of the second category with the Baire property and $Z + Z \subset Z$, then there exists a number $c > 0$

such that $(c, \infty) \subset Z$ or a number $c \leqslant 0$ such that $(-\infty, c) \subset Z$.

Theorem 2. Each at most countable family $\{Z_n\}_{n=1}^{\infty}$ of proper subsets of R^1, fulfilling the following conditions

1^o Z_n has the Baire property,

2^o $Z_n \cap Z_m = \{0\}$, $n \neq m$, $Z_n \neq \{0\}$,

3^o $Z_n + Z_n = Z_n$,

4^o $\bigcup_{n=1} Z_n = R$

consists of only two sets $Z_{n_1} = [0, \infty]$ and $Z_{n_2} = [-\infty, 0]$.

P r o o f. Since $\bigcup_{n=1}^{\infty} Z_n = R$, there exists a set Z_{n_1} of the second category. According to *Lemma 1*, there exists a number $c \leqslant 0$ such that $(-\infty, c) \subset Z_{n_1}$ or a number $c \geqslant 0$ such that $(c, +\infty) \subset Z_{n_1}$.

Suppose that the latter condition is fulfilled. This fact, together with 2^o and 3^o, implies that $(0, \infty) \subset Z_{n_1}$ and also $[0, \infty) \subset Z_{n_1}$. Z_{n_1} is a proper subset of R, so, by 3^o, $Z_{n_1} = [0, \infty)$.

In such a case, $(-\infty, 0] = \bigcup_{n \neq n_1} Z_n$ and there exists a set Z_{n_2} of the second category, containing the set $(-\infty, 0]$. By the same arguments we used previously, we prove that $Z_{n_2} = (-\infty, 0]$.

Theorem 3. Let $\{Z_n\}_{n=1}^{\infty}$ be a countable family of proper subsets of R^k, fulfilling the following conditions

1^o Z_n has the Baire property,

2^o $Z_n \cap Z_m = \{0\}$ for $n \neq m$,

3^o $Z_n + Z_n = Z_n$.

If $\bigcup_{n=1}^{\infty} Z_n = R^k$, then every set Z_n is of the form $Z_n = \mathrm{Int} \bigcap_{l=1}^{\infty} \overline{P}_l \cup Z_o^n$, where $\{P_l\}_{l=1}^{\infty}$ is a countable family of half-spaces each of which is determined by a (k - 1) - dimensional hyperplane passing through the origin and Z_o^n is a set of the first category. Moreover, if $\mathrm{Int}\, Z_n \neq 0$, then $Z_o^n \subset \mathrm{Fr} \bigcap_{l=1} \overline{P}_l$.

P r o o f. Let S be the unit sphere in R^k.
For each point $p \in S$, let

$$L(p) = \{rp : r \in (0,\infty)\}$$

be the open ray passing through p. Let $l(p)$ denote a line passing through p, and $\hat{L}(p) = l(p) - L(p)$. By proving *Theorem 2* in [1], K u c z m a showed that, for each point $p \in S - S_n$, where S_n is a set of the first category in the induced topology of S, a set $Z_n \cap L(p)$ has the Baire property considered as a subset of R. For each point $p \in S$, where $S' = \bigcup_{n=1}^{\infty} S_n$, a family of sets $K_o = \hat{L}(p)$, $K_n = Z_n \cap \overline{L(p)}$, n = 1,2, ..., fulfils assumptions 1^o-3^o of *Theorem 2*, and $\bigcup_{i=0}^{\infty} K_i = R$. Hence the family $\{K_i\}_{i=0}^{\infty}$ consists of two sets $K_{n_1} = \hat{L}(p)$ and $K_{n_2} = \overline{L(p)}$. Then, for each point $p \in S - S'$,

$$Z_n \cap L(p) = \emptyset \quad \text{or} \quad Z_n \cap L(p) = L(p).$$

Let

$$A_n = \{p \in S : Z_n \cap L(p) = L(p)\},$$

$$B_n = \bigcup_{p \in A_n} L(p),$$

$$C_n = \bigcup_{p \in S_n} L(p).$$

The sets of family $\{A_n\}_{n=1}^{\infty}$ are mutually disjont as well as the sets of family $\{B_n\}_{n=1}^{\infty}$. Further, each set C_n, n = 1,2, ..., is a set of the first category.

It is easy to see that

$$(*) \qquad \bigcup_{n=1}^{\infty} B_n \cup \bigcup_{n=1}^{\infty} C_n \cup \{0\} = R^k.$$

Now, we prove that

$$\bigcup_{n=1}^{\infty} C_n \cap Z_{n_o} = C_{n_o} \cap Z_{n_o}.$$

Let $x \in \bigcup_{n=1}^{\infty} C_n \cap Z_{n_o}$; then $x \in Z_{n_o}$ and there exists n_1 such that $x \in C_{n_1}$. If $n_1 = n_o$, then $x \in C_{n_o} \cap Z_{n_o}$. Let $n_1 \neq n_o$. Since $x \in C_{n_1}$ there exists $p \in S_{n_1}$ such that $x \in L(p)$. A set $L(p) \cap Z_{n_o}$ is non-empty because $x \in L(p) \cap Z_{n_o}$. If the set $L(p) \cap Z_{n_o}$ had the Baire property, then it would be true that $L(p) \cap Z_{n_o} = L(p)$, which is not possible because then $L(p) \cap Z_{n_1} = \emptyset$ while $p \in S_{n_1}$. Hence $L(p) \cap Z_{n_o}$ has not the Baire property and so, $p \in S_{n_o}$, $L(p) \subset C_{n_o}$ and, consequently, $x \in C_{n_o} \cap Z_{n_o}$.

By (*) and the previous equation,

$$Z_n = B_n \cup C_n \cap Z_n \cup \{0\}.$$

K u c z m a in [1] proved that B_n is a cone and also a convex set. Since B_n and B_m are disjont and convex, it is possible to seperate them by a $(k-1)$-dimensional hyperplane π_m passing through the origin [2].

Let P_m denote an open half-space in whose closure B_n lies. Hence we obtain

$$B_n \subset \bigcap_{l=1}^{\infty} \overline{P}_l.$$

Further, weshall show that

$$\operatorname{Int} \bigcap_{l=1}^{\infty} \overline{P}_l = \operatorname{Int} B_n = \operatorname{Int} Z_n.$$

First, we shall prove that $\mathrm{Int} \bigcap_{l=1}^{\infty} \overline{P}_l = \mathrm{Int}\, B_n$. Suppose that $\mathrm{Int} \bigcap_{l=1}^{\infty} \overline{P}_l \neq O$ and let $\mathrm{Int} \bigcap_{l=1}^{\infty} P_l - B_n \neq O$. Then there exists $x \in \mathrm{Int} \bigcap_{l=1}^{\infty} \overline{P}_l$, and $x \notin B_n$. There exists a positive number ε such that a ball $K(x, \varepsilon)$ is included in $\mathrm{Int} \bigcap_{l=1}^{\infty} P_l$. Let B_x be a hyperplane seperating a set $\{x\}$ from a set B_n. By P_x we denote a half-space in whose closure $\{x\}$ lies.

We obtain

$$K(x, \varepsilon) \cap P_x \cap B_n = \emptyset$$

and also

$$\emptyset \neq K(x, \varepsilon) \cap P_x \subset \mathrm{Int} \bigcap_{l=1}^{\infty} \overline{P}_l - B_n \subset \bigcap_{l=1}^{\infty} \overline{P}_l - B_n =$$

$$\bigcap_{l=1}^{\infty} \overline{P}_l - \bigcap_{l=1}^{\infty} P_l \cup \bigcap_{l=1}^{\infty} P_l - B_n .$$

It is easy to see that

$$\bigcap_{l=1}^{\infty} \overline{P}_l - \bigcap_{l=1}^{\infty} P_l \subset \bigcap_{l=1}^{\infty} \pi_l$$

and we shall also prove that

$$\bigcap_{l=1}^{\infty} \overline{P}_l - B_n \subset \bigcap_{n=1}^{\infty} C_n .$$

Since $\bigcup_{n=1}^{\infty} C_n \cup \bigcup_{n=1}^{\infty} B_n \cup \{O\} = R^k$, therefore if $x \in \bigcap_{l=1}^{\infty} P_l - B_n$, then $x \in \bigcup_{n=1}^{\infty} C_n$ or $x \in \bigcup_{n=1}^{\infty} B_n$. The latter would be impossible because then both $x \in B_{n_o}$, $n \neq n_o$, and $x \in P_{n_o}$, but from the definition of $\{P_l\}_{l=1}^{\infty}$ we have $P_{n_o} \cap B_n = \emptyset$. Thus we obtain that a set $\mathrm{Int} \bigcap_{l=1}^{\infty} \overline{P}_l - B_n$, which has non-empty interior is a subset of the union of two sets of the first category $\bigcup_{l=1}^{\infty} \pi_l$

and $\bigcup_{n=1}^{\infty} C_n$, which is a contradiction. Hence $\mathrm{Int} \bigcap_{l=1}^{\infty} \overline{P}_l = B_n$ and $\mathrm{Int} \bigcap_{l=1}^{\infty} \overline{P}_l = \mathrm{Int}\, B_n$.

Proving the next equality $\mathrm{Int}\, B_n = \mathrm{Int}\, Z_n$, it suffices to show that $\mathrm{Int}\, Z_n \subset B_n$. If $\mathrm{Int}\, Z_n = \emptyset$, then also $\mathrm{Int}\, B_n = \emptyset$. Suppose that $x \in \mathrm{Int}\, Z_n$. Then there exist a ball $K(x, \varepsilon) \subset \mathrm{Int}\, Z_n \subset Z_n$ and $a, b, \in R^k$, such that the segment $[a, b] \subset L(x) \cap Z_n$. It is easy to verify that a set $[na, nb] \subset L(x) \cap Z_n$ for $n = 1, 2, \ldots$ and, in consequence, each element $z \subset L(x)$ belongs to the set $L(x) \cap Z_n$. Therefore, $L(x) \cap Z_n = L(x)$ and $L(x) \subset B_n$. The equality below follows from $*$:

$$Z_n = \mathrm{Int}\, B_n \cup B_n - \mathrm{Int}\, B_n \cup C_n \cap Z_n \cup \{0\}.$$

Let us denote

$$Z_o^n = B_n - \mathrm{Int}\, B_n \cup C_n \cap Z_n \cup \{0\}.$$

Since $B_n - \mathrm{Int}\, B_n \subset \bigcap_{l=1}^{\infty} \overline{P}_l - \mathrm{Int} \bigcap_{l=1}^{\infty} \overline{P}_l$, therefore Z_o^n is a set of the first category in R^k. Thus $Z_n = \mathrm{Int} \bigcap_{l=1}^{\infty} \overline{P}_l \cup Z_o^n$. Moreover, if $\mathrm{Int}\, Z_n \neq \emptyset$, then $Z_o^n \subset \mathrm{Fr} \bigcap_{l=1}^{\infty} \overline{P}_l$.

It is sufficient to prove that the set $C_n \cap Z_n$ is a subset of $\bigcap_{l=1}^{\infty} \overline{P}_l - \mathrm{Int} \bigcap_{l=1}^{\infty} \overline{P}_l = \mathrm{Fr} \bigcap_{l=1}^{\infty} \overline{P}_l$. Let $x \in C_n \cap Z_n$. It is clear that $x \notin \mathrm{Int} \bigcap_{l=1}^{\infty} \overline{P}_l$ because $\mathrm{Int} \bigcap_{l=1}^{\infty} \overline{P}_l$ $\mathrm{Int}\, B_n$. Therefore, it is sufficient to show that $x \in \overline{P}_l$ for each $l = 1, 2, \ldots$ Suppose that $x \notin P_{l_o}$. Then $x \in P'_{l_o} = R^k - \overline{P}_{l_o}$ and there exists a ball $K(x, \varepsilon) \subset P'_{l_o}$. Since $\mathrm{Int}\, Z_n \neq \emptyset$ and $\mathrm{Int}\, Z_n = \mathrm{Int} \bigcap_{l=1}^{\infty} \overline{P}_l$, therefore, for every positive number δ, $K(0, \delta) \cap \mathrm{Int} \bigcap_{l=1}^{\infty} \overline{P}_l$ is a non-empty open set. Let $y \in K(0,\varepsilon) \cap \mathrm{Int} \bigcap_{l=1}^{\infty} \overline{P}_l$;

then there exists a ball $K(y, \eta)$ such that $K(y, \eta) \subset K(0, \varepsilon) \cap \cap \operatorname{Int} \bigcap_{l=1}^{\infty} \bar{F}_l \subset K(0, \varepsilon)$. Obviously, $K(y, \eta) \subset Z_n$ and also $K(y, \eta) + + x \subset Z_n$, $K(y, \eta) + x \subset K(x, \varepsilon)$, but this is impossible, since the set $K(x, \varepsilon)$ is disjont from $\operatorname{Int} Z_n$. If, instead of a countable family, we consider two sets Z_1 and $Z_2 = R^k - Z_1$, which fulfil 1^o-4^o, then we obtain *Theorem 2* of [1]. It turns out that the Baire property cannot be omitted and the countable family cannot be replaced by an uncountable one.

Example 1. There exist two sets Z_1, $Z_2 \subset R$, $Z_1, Z_2 \neq R$ fulfilling the conditions

1^o $0 \notin Z_1$, $0 \in Z_2$, $Z_1 \cap Z_2 = \{0\}$,

2^o $Z_1 + Z_1 = Z_1$, $Z_2 + Z_2 = Z_2$,

3^o $Z_1 \cup Z_2 = R$,

but $Z_1 \neq (-\infty, 0]$, $Z_1 \neq [0, \infty)$ and then, $Z_2 \neq (-\infty, 0]$, $Z_2 \neq \neq [0, +\infty)$.

Let $H = \{h_i\}_{i<\Omega}$ denote the Hamel basis in the vector space R over the field of rational numbers Q. Each element $x \in R$, $x \neq 0$, has the unique representation

(**) $$x = a_1 h_{i_1} + \dots + a_n h_{i_n}, \quad i_1 < i_2 < \dots < i_n$$

$$a_1, \dots, a_n \in Q.$$

Let Z_1 denote the set of all points for which a_n is a positive number in representation **. Moreover, let $0 \in Z_1$. It is easy to verify that $Z_1 + Z_1 = Z_1$. Let $Z_2 = R - Z_1 \cup \{0\}$. Obviously, $Z_2 + Z_2 = Z_2$, $Z_1 \cap Z_2 = \{0\}$ and $Z_1 \cup Z_2 = R$. The set Z_1 does not contain the non-degenerate interval because Z_1 is disjont from the set $\{x : x = ah_1 - h_2,\ h_1 < h_2,\ a \in Q\}$ which is dense on the real line R. Similar arguments apply to Z_2. Hence the sets Z_1 and Z_2 have not the form described in *Theorem 3*. The sets Z_1 and Z_2 have not the Baire property because Z_1 or Z_2 is a set of the second category and, by the P i c c a r d theorem [3], we should obtain that Z_1 or Z_2 contains an open segment.

Example 2. There exists an uncountable family of sets $\{Z_t\}_{t \in T} \subset R$, such that

1° $Z_t \cap Z_s = \{0\}$ $t \neq s$, $Z_t \neq 0$,

2° Z_t has the Baire property,

3° $Z_t + Z_t = Z_t$,

4° $\bigcup_{t \in T} Z_t = R$.

Let H be the Hamel basis. Let $Z_{(h_1, \ldots, h_n)} = \{x : x = a_1h_1 + \ldots + a_nh_n,\ h_1 \ldots h_n \in H,$

$$a_1 \neq 0, \ldots, a_n \neq 0\},$$

and let $0 \in Z_{(h_1, \ldots, h_n)}$.

$\bigcup_{n \in N} Z_{(h_1, \ldots, h_n)} = R$ and $Z_{(h_1, \ldots, h_n)}$ is a countable set, so it

$$h_1, \ldots, h_n \in H$$

has the Baire property, and also

$$Z_{(h_1, \ldots, h_n)} + Z_{(h_1, \ldots, h_n)} = Z_{(h_1, \ldots, h_n)}.$$

By the properties of the Hamel basis, condition 1° is satisfied and $h_1 + \ldots + h_n \in Z_{(h_1, \ldots, h_n)}$, which implies that $Z_{(h_1, \ldots, h_n)} \neq 0$.

Simultaneously, the set $T = \{(h_1, \ldots, h_n) : n \in N,\ h_1, \ldots, h_n \in H\}$ is uncountable.

REFERENCES

[1] M. Kuczma, *On a theorem of Roman Ger*, Prace Nauk. Uniw. Śląsk. No 87.

[2] K. Kuratowski, *Topologie*, vol. I (Quatrième edition), Monogr. Mat. 20, Warsaw 1958.

[3] S. Piccard, *Sur des ensembles parfaits*, Paris 1942.

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PEWNE WŁASNOŚCI PODZBIORÓW PRZESTRZENI R^k O WŁASNOŚCI BAIRE'A

W pracy uzyskane jest uogólnienie pewnego twierdzenia Kuczmy, które sformułować możemy w następujący sposób:

Twierdzenie. Niech $\{Z_n\}_{n=1}^{\infty}$ będzie przeliczalną rodziną właściwych podzbiorów przestrzeni R^k, spełniającą następujące warunki:

1° Z_n ma własność Baire'a,

2° $Z_n \cap Z_m = \{\emptyset\}$ dla $n \neq m$,

3° $Z_n + Z_n = Z_n$.

Jeśli $\bigcup_{n=1}^{\infty} Z_n = R^k$, wtedy każdy zbiór Z_n jest postaci $Z_n = \operatorname{Int} \bigcap_{l=1}^{\infty} \overline{P}_l \cup Z_o^n$, gdzie $\{P_l\}_{l=1}^{\infty}$ jest przeliczalną rodziną półprzestrzeni, z których każda wyznaczona jest przez $(k-1)$ - wymiarową hiperpłaszczyzną przechodzącą przez początek układu współrzędnych i Z_o^n jest zbiorem pierwszej kategorii. Jeśli $\operatorname{Int} Z_n \neq \emptyset$, to $Z_o^n \subset \operatorname{Fr} \bigcap_{l=1}^{\infty} \overline{P}_l$.

W pracy podane są również dwa przykłady ilustrujące konieczność założenia własności Baire'a i przeliczalności rodziny zbiorów stanowiących pewien rozkład przestrzeni R^k.