A C T A U N I V E R S I T A T I S L O D Z I E N S I S
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ON THE REPRESENTATION OF THE DENSITY FUNCTIONS GAMMA TYPE

To Professor Lech Włodarski on His 80th birthday

In this paper we present the random variable X of gamma type as infinite (or finite) product of independent random variables X_k, $k \in N$, in the following sense:

$$X \overset{st}{=} \prod_{k=1}^{\infty} X_k.$$

Let the random variable X have three-parameters gamma distribution with the density given by the formula

(1)
$$f(x|\Theta,\, \lambda,\, p) = \frac{\lambda}{\Theta^{p/\lambda}\Gamma^{-1}\left(\frac{p}{\lambda}\right)}\, x^{p-1} \exp\left(-\frac{x^{\lambda}}{\Theta}\right), \quad x \geq 0,\ p, \lambda, \Theta > 0,$$

where $\Theta^{1/\lambda}$ is the scale parameter and p, λ are the shape-parameter.

In the present note it will be used the Mellin transform (2) of the function (1)

$$\begin{aligned} M(s) &= M\,[f)x|\Theta,\, \lambda,\, p),\, s] = EX^s \\ &= \Theta^{s/\lambda}\Gamma\left(\frac{p+s}{\lambda}\right)\Gamma^{-1}\left(\frac{p}{\lambda}\right), \quad \operatorname{Re} s > -p, \end{aligned} \tag{2}$$

where s denote a complex variable: [1], [2]: recall that Sneddon [3], Dytkin and Prudnikow [4] have defined the Mellin transform $M(s)$ as EX^{s-1}.

By applying the Knar formula [5.8, p. 324]

$$\Gamma(x+1) = 2^{2x} \prod_{k=1}^{\infty} \frac{\gamma\left(\frac{1}{2} + \frac{x}{2^k}\right)}{\Gamma\left(\frac{1}{2}\right)}, \quad \operatorname{Re} x > -1,$$

into (2) we obtain

$$M(s) = (4\Theta)^{s/\lambda} \prod_{k=1}^{\infty} \frac{\gamma\left(\frac{1}{2} + \frac{p-\lambda+s}{2^k \lambda}\right)}{\gamma\left(\frac{1}{2} + \frac{p-\lambda}{2^k\lambda}\right)}, \quad \operatorname{Re} > -p.$$

The last formula can be written as

$$(3) \qquad M(s) = \prod_{k=1}^{\infty} \frac{(4\Theta)^{s/\lambda 2^k}\Gamma\left(\frac{1}{2} + \frac{p-\lambda+s}{2^k\lambda}\right)}{\Gamma\left(\frac{1}{2} + \frac{p-\lambda}{2^k\lambda}\right)}, \quad \operatorname{Re} s > -p.$$

Let us denote k-factor occuring in the right hand side of (3) by $g_k(s)$. Next, we apply the inverse Mellin transform for every factor $g_k(s)$; the result of this transform is denoted by $f_k(x|\Theta, \lambda, p)$:

$$(4) \qquad f_k(x|\Theta, \lambda, p) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} x^{-s-1} g_k(s)\, ds, \quad \operatorname{Re} s > -p.$$

Putting $c = 0$ in view $\operatorname{Re} s > -p$, we get

$$(5) \quad f_k(x|\Theta, \lambda, p) = \frac{2^k\lambda}{\Gamma\left(\frac{1}{2} + \frac{p-\lambda}{2^k\lambda}\right)(4\Theta)^{\frac{1}{2}+\frac{p-\lambda}{2^k\lambda}}} x^{(2^{k-1}-1)\lambda+p-1} \exp\left(-\frac{x^{2^\lambda}}{2\Theta}\right), \quad x \geq 0.$$

It is easy to prove that the condition $\int_0^\infty f_k(x|\Theta, \lambda, p)dx = 1$ is satisfied for any $k \in \mathbb{N}$. In view of the inequality $f_k(x|\Theta, \lambda, p) \geq 0$

this implies that the formula (5) defines the density of probability of the random variable X_k in the interval $[0,\infty)$. In consequaence every factor $g_k(s)$, $k\in\mathbb{N}$ of the infinity product (3) is the Mellin transform of the random variable X_k with the density given by (5).

Now we shall show that $\prod_1^\infty g_k(s)$ is the Mellin transform of $\prod_1^\infty f_k(x|\Theta,\lambda,p)$. To do this we find an estimate of an absolute value of any factor of (3). In fact setting $\operatorname{Re} s=c=0$ according to $\operatorname{Re} s=c=0$ we get

$$\left|\frac{(4\Theta)^{s/2^k\lambda}\Gamma\left(\frac{1}{2}+\frac{p-\lambda+s}{2^k\lambda}\right)}{\Gamma\left(\frac{1}{2}+\frac{p-\lambda}{2^k\lambda}\right)}\right|\leq\frac{(4\Theta)^{\operatorname{Re} s/2^k\lambda}\Gamma\left(\frac{1}{2}+\frac{p-\lambda+\operatorname{Re} s}{2^k\lambda}\right)}{\Gamma\left(\frac{1}{2}+\frac{p-\lambda}{2^k\lambda}\right)}\leq 1.$$

Let us denote $M_n((s)=\prod_{k=1}^n g_k(s)$. Besides, for any $n\in\mathbb{N}$ $|M_n(s)|\leq|M_1(s)|$ and $M_1(s)$ is an absolute integrable function. From the above and in view of the Lebesgue dominated convergence theorem [6. Th. 1.34] and (3) we obtain successively

$$\lim_{n\to\infty}\int_{-i\infty}^{i\infty}x^{-s-1}\left(M(s)-\lim_{n\to\infty}M_n(s)\right)ds=0.$$

Thus, we have proved the following

Theorem. *The density of the three-parameter gamma distributuin of the form (1) is equal to the density of the infinite product* $\prod_1^\infty X_k$, *where the density* $f_k(x|\Theta,\lambda,p)$ *of* X_k *are independent and defined by the formula (5), the scale parameter being* $(4\Theta)^{1/2^k\lambda}$ *respectively, what can be formally denoted as*

$$X\stackrel{\mathrm{st}}{=}\prod_{k=1}^{\infty}X_k.$$

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O PEWNEJ REPREZENTACJI FUNKCJI GĘSTOŚCI TYPU GAMMA

Przed ca 30 laty ukazało się wiele prac dotyczących wyznaczania gęstości skończonego iloczynu zmiennych losowych niezależnych o rozkładach Beta, Gamma, Normalnym, Bessela, w tym prace pracowników naukowych Politechniki Łódzkiej: Śródki, Podolskiego i Krysickiego.

Prezentowana praca stawia sobie jako cel: cel przeciwny, którym jest przedstawienie zmiennej losowej X typu Gamma jako nieskończonego (bądź skończonego) iloczynu zmiennych losowych niezależnych X_k, $k \in \mathbb{N}$, s sensie stichastycznym tzn. w sensie równości gęstości $X \overset{\text{stoch}}{=} \prod_1^\infty X_k$.

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