ACTA UNIVERSITATIS LODZIENSIS
FOLIA MATHEMATICA 8, 1996

Ryszard Jajte

ON SUMMABILITY METHODS GENERATED BY R–INTEGRABLE FUNCTIONS

To Professor Lech Włodarski on His 80th birthday

A class of summability methods being deformations of ordinary arithmetical means is discussed.

Let φ be a function defined on the interval $[0,1]$ and integrable in the sense of Riemann with $\int_0^1 \varphi(t)dt = 1$.

We adopt the following definition

Definition 1. We say that a sequence $x = \{\xi_k\}$ is summable by the method (φ) to a number α iff

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \varphi\Big(\frac{k}{n}\Big)\xi_k = \alpha.$$

We then write $(\varphi)-\lim_{k\to\infty} \xi_k = \alpha$. Obviously, each method (φ) is regular (permanent), i.e. $\xi_n \to \xi$ always implies $(\varphi)-\lim_{n\to\infty} \xi_n = \xi$. We shall say that the method (φ) is generated by φ. The class of all such methods will be denoted by (R).

Theorem 1. *Every Cesàro method (C,r) (with $r \geq 1$) belongs to (R).*

Proof. Putting $\varphi \equiv 1$ we get $(C,1)$. Assume $r > 1$. Then the method (C,r) is given by the matrix

$$C_{n,k}^{(r)} = \begin{cases} \dfrac{rn!\Gamma(n-k+r)}{(n-k)!\Gamma(n+r+1)}, & 0 \leq k \leq n, \\ 0 & k > n. \end{cases}$$

We have to show that there exists an R-integrable function φ_r with $\int_0^1 \varphi_r(t)dt = 1$ and such that

$$\varphi_r\left(\frac{k}{n}\right) = nC_{n-1,k}^{(r)} \qquad (k = 0, \dots, n-1; \ \ n = 1, 2, \dots).$$

Let us notice that if $k_s < n_s$, $n_s \to \infty$ and $k_s/n_s \to x$ as $s \to \infty$, then

$$\lim_{s\to\infty} n_s C_{n_s-1,k_s}^{(r)} = \lim_{s\to\infty} \frac{rn_s!\Gamma(n_s-k_s-1+r)}{(n_s-k_s-1)!\Gamma(n_s+r)} = r(1-x)^{r-1}.$$

We put

$$\varphi_r(x) = \begin{cases} nC_{n-1,k}^{(r)}, & x = \frac{k}{n}, \ \ k = 0, \dots, n-1; \ \ n = 1, 2, \dots, \\ r(1-x)^{r-1} & \text{elsewhere} \end{cases}$$

The function φ_r is bounded in $[0,1]$ and continuous almost everywhere, so R-integrable. Obviously, φ_r generates the method (C,r).

Definition 2. We say that the method of Toeplitz $(a_{n,k})$ belongs to (K) iff

$$a_{n,k} = (1-x_n)x_n^k f(x_n^k), \qquad k, n = 0, 1, \dots,$$

where f is R-integrable on $[0,1]$ with $\int_0^1 f(t)dt = 1$, $0 < x_n < 1$, $x_n \to 1$.

Theorem 2 (Karamata [1]). *If a sequence $\{\xi_k\}$ is bounded and summable by the Abel method to ξ then, for every function f, R-integrable on the interval [0,1], we have*

$$\lim_{t\to 1-}(1-t)\sum_{k=0}^{\infty} t^k f(t^k)\xi_k = \xi\int_0^1 f(u)du.$$

From the above theorem it follows that all methods in (K) are regular and that, for bounded sequences, they are not weaker than Cesàro means and consistent with them.

Theorem 3. $(R)\subset(K)$.

Proof. Let a method $(\varphi)\in(R)$ be given. Let us put $x_n = 1-1/n$ $(n = 1,2,\dots)$. We shall show first that there exists a function g bounded, R-integrable on the interval $e^{-1}\le x\le 1$ and such that $g[(1-1/n)^k] = \varphi(k/n)$, for $k = 0,\dots,n-1$; $n = 1,2,\dots$. To this end, let us remark that $k<n$ iff $(1-1/n)^k > e^{-1}$ and that the sequence $\beta_s = (1-1/n_s)^{k_s}$ $(k_s<n_s)$ is convergent iff the limit $\lim_{s\to\infty} k_s/n_s$ exists.

Let us put

$$g(x)=\begin{cases}\varphi(k/n) & \text{for } x=(1-1/n)^k,\ k=0,\dots,n-1;\\ & \qquad n=1,2,\dots,\\ \varphi(-\log x), & \text{elsewhere}\end{cases}$$

Clearly, the function g is bounded and continuous almost everywhere.

We put

$$f(x)=\begin{cases}0, & 0\le x\le e^{-1},\\ g(x)/x & e^{-1}\le x\le 1.\end{cases}$$

It is easy to check that the function f and the sequence $x_n = 1-1/n$ determine a method from (K), identical with (φ).

From the above theorem it follows that every method $(\varphi)\in(R)$ is not weaker than $(C,1)$, for bounded sequences. On the other hand,

K. Knopp [2] proved that there exists a bounded sequence summable $(C,1)$ but not summable by any Euler method (E,p). This implies that the Euler methods (E,p) do not belong to (R). It is worth noting here that there exist two equivalent Hausdorff methods such that one of them belongs to (R) and the second one does not. Namely, the Cesàro method $(C,2) \in (R)$ but the Hölder method $(H,2) \notin (R)$. Indeed, the $(H,2)$-transform of $\{\xi_k\}$ is of the form $n^{-1}\sum_{k=1}^{n}\left(\sum_{\nu=k}^{n}\nu^{-1}\right)\xi_k$, so $na_{n,1} = \sum_{\nu=1}^{n}\nu^{-1} \to \infty$, which is impossible for the method from (R).

Now, we are going to show a fact which is a little bit paradoxical. Namely, from the next theorem it will follow that there exist in (R) two nonequivalent methods generated by two characteristic functions (of sets of measure one).

Theorem 4. *Let $\mathcal{C}$ denote the Cantor set on the interval $[0,1]$ and let φ be the indicator of the complement of $\mathcal{C}$. Then there exists a sequence $\{\xi_k\}$ (unbounded!) which is $(C,1)$ summable and is not (φ)-summable.*

Proof. From the geometric construction of the Cantor set we can deduce the following inequality

$$3^{-n}\sum_{\substack{1\le k\le 3^n\\ k3^{-n}\in\mathcal{C}}}\sqrt{k} > M\Big(\frac{2}{\sqrt{3}}\Big), \qquad M>0.$$

We divide the set $\mathbb{N}$ of positive integers into two parts. Namely, a positive integer k belongs to Ω_1 iff there exists a positive integer n such that $k3^{-n}\in\mathcal{C}$. Let $\Omega_2 = \mathbb{N}-\Omega_1$. We arrange Ω_1 into increasing sequence:

$$k_1 < k_2 \dots .$$

We do the same with Ω_2:

$$l_1 < l_2 \dots .$$

Then we set

$$\xi_\nu = \begin{cases} \sqrt{\nu} & \text{for } \nu\in\Omega_1,\\ -\sqrt{k_s} & \text{for } \nu = l_s,\ \ l_s > k_s,\\ 0 & \text{for } \nu = l_s,\ \ l_s < k_s. \end{cases}$$

Let $n \in \mathbb{N}$, and let $k_1 < k_2 \ldots < k_{p(n)}$ be all numbers from Ω_1 not greater than n. Similarly, let $l'_1 < l'_2 \ldots < l'_{q(n)}$ be all numbers from Ω_2 not greater than n and such that $l_s > k_s$. Then we have $0 \le p_n - q_n \le 6$ $(n = 1, 2, \ldots)$. The sequence $\{\xi_\nu\}$ defined above is $(C, 1)$-summable to zero. Indeed,

$$\left|\frac{1}{n}\sum_{\nu=1}^{n}\xi_\nu\right| = \left|\frac{1}{n}\sum_{\substack{1\le\nu\le n\\ \nu\in\Omega_1}}\xi_\nu + \frac{1}{n}\sum_{\substack{1\le\nu\le n\\ \nu\in\Omega_2}}\xi_\nu\right|$$
$$= \frac{1}{n}\left|\sum_{s=1}^{p_n}\sqrt{k_s} - \sum_{s=1}^{q_n}\sqrt{k_s}\right| \le \frac{1}{n}6\sqrt{n} \to 0.$$

On the other hand,

$$\frac{1}{n}\sum_{k=1}^{n}\varphi\left(\frac{k}{n}\right)\xi_k = \frac{1}{n}\sum_{k=1}^{n}\xi_k - \frac{1}{n}\sum_{\substack{1\le k\le n\\ k/n\in C}}\xi_k.$$

We already know that $\frac{1}{n}\sum_{k=1}^{n}\xi_k \to 0$. But

$$\frac{1}{3^n}\sum_{\substack{1\le k\le 3^n\\ k3^{-n}\in C}}\xi_k = \frac{1}{3^n}\sum_{\substack{1\le k\le 3^n\\ k3^{-n}\in C}}\sqrt{k} \to \infty.$$

Thus the sequence (ξ_ν) is not summable (φ), which ends the proof.

References

[1] J. Karamata, *Über die Hardy–Littlewoodschen Umkehrungen des Abelschen Stetigkeitssatzes*, M.Z. **32** (1930), 319–320.

[2] K. Knopp, *Über das Summirungsverfahren*, M.Z. **15** (1922), 226–253.

Ryszard Jajte

O METODACH SUMOWALNOŚCI GENEROWANYCH PRZEZ FUNKCJE R-CAŁKOWALNE

W pracy omówiono pewną klasę metod sumowalności wynikającą ze zniekształcenia średnich arytmetycznych funkcjami całkowalnymi w sensie Riemanna.

Institute of Mathematics
Łódź University
ul. S. Banacha 22, 90–213 Łódź, Poland