A C T A   U N I V E R S I T A T I S   L O D Z I E N S I S

FOLIA MATHEMATICA 9, 1997

Archil Kipiani

ONE ABSTRACT CHARACTERIZATION OF INTERVALS OF CARDINAL NUMBERS

We give, in the language of functions and automorphisms of binary relations, a characterization of the interval of cardinal numbers $[\omega_0, 2^{\omega_0}]$ and of the interval of cardinal numbers $(\kappa, \lambda]$, where κ and λ are uncountable beths.

In this paper we shall use the standard set theoretical and graph theoretical terminology. The cardinality of a set X is denoted by $|X|$. By $P(X)$ we denote the family of all subsets of the set X. By $\omega = \omega_0$ we denote the first infinite ordinal number. Simultaneously, ω_0 denotes the cardinality of the ordinal number ω. We shall identify any ordinal number ζ with the set of all those ordinal numbers which are strictly smaller than ζ, and we shall identify some functions with their graphs.

For any ordinal number ξ, the ξ-th cardinal number from the hierarchy of beths is denoted by a_ξ. Namely, we put by recursion

$$a_0 = \omega_0,$$

$$a_{\xi+1} = 2^{a_\xi},$$

and, for a limit ordinal number ζ,

$$a_\zeta = \sum\{a_\xi : \xi < \zeta\}.$$

If κ and λ are arbitrary cardinal numbers, then by $(\kappa, \lambda]$ we denote the set

$$\{\mu : \mu \text{ is a cardinal number and } \kappa < \mu \leq \lambda\}.$$

In works [1], [2] and [5] some combinatorial properties of isomorphisms and automorphisms of binary relations (such as graphs, partitions, partial orders and trees) are considered.

In this paper we will use the corresponding combinatorial properties of trees for finding a characterization of various intervals of cardinal numbers.

By an oriented tree we shall mean a tree with a root. In addition, we shall assume, in our further considerations, that a tree is oriented from the root. We shall identify a tree T with the corresponding set of ordered pairs and by $V(T)$ we shall denote the set of all vertices of T.

We announced in [3] the following theorem.

Theorem 1. *Let E be a set such that $|E| \geq 3$. Then the following two conditions are equivalent:*

1) $\omega_0 \leq |E| \leq 2^{\omega_0}$.

2) *there exists a function $f : E \to E$ such that*

(a) *there exists $x_0 \in E$ such that $|f^{-1}(x_0)| = |E|$ and, for every $x \in E \setminus \{x_0\}$, we have $|f^{-1}(x)| \leq 2$,*

(b) *the group of all automorphisms of the structure (E, f) is trivial.*

Proof. Suppose that $\omega_0 \leq |E| \leq 2^{\omega_0}$. Let X be a s qqubset of $P(\omega)$ of the cardinality $|E|$. We define the following tree Y:

$$Y = \{((S, n), (S, n+1)) : S \in X \ \&\ n \in \omega\} \cup$$

$$\{((S,n),(S,n,0)) : S \in X \;\&\; n \in S\} \cup \{(0,(S,0)) : S \in X\}.$$

It is clear that the root of the tree Y has $|E|$ successors and any other vertex of Y has at most 2 successors.

The function

$$Y^{-1} \cup \{(0,0)\} : V(Y) \to V(Y)$$

has the required properties and, since $|V(Y)| = |E|$, the proof of implication (1) $\Rightarrow$ (2) is done.

Suppose now that condition (2) holds. Let

$$f : E \to E$$

be a function which has the properties (a) and (b) and suppose that

$$\omega_0 \leq |E| \leq 2^{\omega_0}$$

does not hold. There are two possibilities:

$$|E| < \omega_0 \;\text{ or }\; |E| > 2^{\omega_0}.$$

Since $|E| \geq 3$, the possibility $|E| < \omega_0$ contradicts the conjunction of the properties (a) and (b).

Suppose now that $|E| > 2^{\omega_0}$ holds and $x_0 \in E$ is such an element that

$$|f^{-1}(x_0)| = |E|.$$

Let

$$Z = f^{-1}(x_0) \setminus \{f^k(x_0) : k \in \omega\}.$$

It is clear that the set Z has the cardinality $|E|$. For every element $z \in Z$, let T_z be a tree given by the formula:

$$T_z = \{(x, f(x)) : x \in E \;\&\; (\exists k \in \omega \setminus \{0\})(f^k(x) = z)\}.$$

It is clear that $|V(T_z)| \leq \omega_0$ for every $z \in Z$. Hence, there are at most 2^{ω_0} many non-isomorphic trees in the family $\{T_z : z \in Z\}$. Thus, there are two distinct elements z_1 and z_2 in the set Z such that the trees T_{z_1} and T_{z_2} are isomorphic. From this fact we deduce that the elements z_1 and z_2 are similar in the graph

$$\{(x, f(x)) : x \in E\}$$

and thus there exists a nontrivial automorphism of the structure (E, f). Hence, Theorem 1 is proved.

In the characterization of intervals of the form $(a_\xi, a_{\xi+1}]$ we shall apply the oriented tree T from [4], which was used there for the solution of one of Ulam's problems and one problem about uniformization.

Let $R_0(E)$ denote relation (2) from Theorem 1 and, for any nonzero ordinal number ξ, let $R_\xi(E)$ denote the following property of the set E:

(i) $(\forall\zeta)(\zeta < \xi \Rightarrow \neg R_\zeta(E))$,

(ii) $(\exists f)(f : E \to E \;\&\; (\exists x_0 \in E)(|f^{-1}(x_0)| = |E| \;\&\; (\forall y \in E \setminus \{x_0\})(|f^{-1}(y)| \in \omega \vee (\exists\zeta < \xi)(R_\zeta(f^{-1}(y))))) \;\&$ the structure (E, f) has no nontrivial automorphism).

Theorem 2. *Let $|E| \geq 3$ and let ξ be a nonzero ordinal number. Then*

$$|E| \in (a_\xi, a_{\xi+1}] \Leftrightarrow R_\xi(E).$$

Proof. We shall prove the theorem by transfinite induction on ξ. At the beginning we shall prove that

$$|E| \in (a_1, a_2] \Leftrightarrow R_1(E).$$

Let us remark that the inductive step is very similar to the proof of this equivalence.

Suppose that $|E| \in (a_1, a_2]$. Then, by Theorem 1, we get $\neg R_0(E)$. We shall show now that condition (ii) holds for the case $\xi = 1$.

Let ω_α be the initial ordinal number of cardinality a_1. For any $n \in \omega \setminus \{0\}$, we define:

$$A_1 = \{(0, \omega^{\xi_1}) : \omega_\alpha > \xi_1\},$$

$$A_{n+1} = \{(\omega^{\xi_1} + \ldots + \omega^{\xi_n}, \omega^{\xi_1} + \ldots + \omega^{\xi_n} + \omega^{\xi_{n+1}}) : \omega_\alpha > \xi_1 \geq \xi_2 \geq \ldots \geq \xi_{n+1}\}.$$

Let T be an oriented tree defined by the formula

$$T = \cup\{A_n : n \in \omega \setminus \{0\}\}.$$

The tree T has the following properties:

1) $(T^{-1}) \cup \{(0,0)\}$ is the graph of a function,

2) every vertex of the tree T which is not the root (i.e. which is not equal to 0) has less than a_1 successors and the root has precisely a_1 successors,

3) T is a rigid tree (i.e. the oriented tree T has a trivial group of automorphisms).

The proof of properties 1) and 2) is easy. The property 3) of the tree T is proved in [4].

Let $(M_i)_{i \in I}$ be a one-to-one enumeration of all subsets of ω_α. For every $i \in I$, we define the following trees:

$$T_{M_i} = \{((\xi, M_i), (\eta, M_i)) : (\xi, \eta) \in T\}$$

$$\cup \{((\xi, M_i), (\xi, M_i, 0)) : \xi \in M_i\}.$$

It is easy to see that, for every $i \in I$, the tree T_{M_i} is rigid. Moreover, if $i, j \in I$ and $i \neq j$, then the sets $V(T_{M_i})$ and $V(T_{M_j})$ are disjoint and the trees T_{M_i} and T_{M_j} are non-isomorphic.

Suppose now that J is a subset of I of cardinality $|E| \in (a_1, a_2]$. It is easy to show that the tree

$$S = \cup\{T_{M_i} : i \in J\} \cup \{(0, (0, M_i)) : i \in J\}$$

has properties analogical to 1), 2) and 3), where in the property 2) the cardinal number a_1 is replaced by the cardinal number $|E|$.

The function

$$(S^{-1}) \cup \{(0,0)\} : V(S) \to V(S)$$

has the required properties and, since the set $V(S)$ of all vertices of the tree S has cardinality $|E|$, we get the required function

$$f : E \to E.$$

Hence, the implication

$$|E| \in (a_1, a_2] \Rightarrow R_1(E)$$

is proved.

The proof of the opposite implication is analogical to the proof of the implication

$$R_0(E) \Rightarrow |E| \in [a_0, a_1]$$

from Theorem 1, but it is necessary to use Theorem 1 in this proof.

Corollary. *For any two nonzero ordinal numbers ξ and η, the following equivalence holds:*

$$|E| \in (a_\xi, a_\eta] \Leftrightarrow (\exists\zeta)(\xi \leq \zeta < \eta \ \& \ R_\zeta(E)).$$

References

[1]. A.B.Kharazishvili, *Elements of Combinatorial Theory of Infinite Sets*, Izd. Tbil. Gos. Univ., Tbilisi, 1981, (in Russian).

[2]. A.Kipiani, *Some combinatorial problems connected with product-isomorphisms of binary relations*, Acta Universitatis Carolinae, Mathematica et Physica **29** no. 2 (1988).

[3]. A.Kipiani, *On one uniform subset in $\omega_\alpha \times \omega_\alpha$*, Bulletin of the Academy of Sciences of the Georgian SSR **135** no. 2 (1989), (in Russian).

[4]. A.Kipiani, *Uniform sets and isomorphisms of trees*, Mathematical Institute, University of Wroclaw, Preprint no. 107 (1989).

[5]. S.Ulam, *A Collection of Mathematical Problems*, Interscience Publishers, New York, 1960.

Archil Kipiani

O ABSTRAKCYJNEJ CHARAKTERYZACJI PRZEDZIAŁÓW LICZB KARDYNALNYCH

W pracy przedstawiona została algebraiczna charakteryzacja, w języku funkcji i automorfizmów relacji binarnych, przedziałów liczb kardinalnych.

Institute of Applied Mathematics
University of Tbilisi
University Str. 2, 380043 Tbilisi 43, Georgia