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ON ALMOST RIGID MATHEMATICAL STRUCTURES

We discuss a concept of an almost rigid mathematical structure and a concept of an almost rigid mathematical structure in the strong sense. We compare these two concepts with the usual notion of a rigid mathematical structure. We also consider an application of the introduced concepts to the theory of linearly ordered sets.

Let Σ be a type of a mathematical structure in the usual sense of N.Bourbaki (see [1]). For example, Σ may be the type of a topological structure, the type of a structure of a measurable space, the type of an order structure and of many others.

Suppose that our type Σ satisfies the following two conditions:

1) for the class of all structures of this type, a class of morphisms (homomorphisms) is defined in such a way that we have a category in the standard algebraic sense;

2) if E is a basic set, S is a structure of the type Σ defined on E, and X is an arbitrary subset of E, then there exists a structure S_X of the same type Σ such that S_X is defined on X and is induced by the original structure S.

Condition 2) can be called a hereditarility property of the given structure type Σ. Notice that topologies, measurable spaces and relation structures satisfy condition 2). There are also many other structures for which this condition is fulfilled.

Let E be again a basic set and let S be some structure of the type Σ defined on E. Let us recall that the structure S is rigid if the group of all automorphisms of S is a one-element set. In other words, the structure S is rigid if and only if the identity transformation of the basic set E is a unique automorphism of S.

In a more general situation, we say that a structure S on E is almost rigid if, for every automorphism $f : (E, S) \to (E, S)$ of this structure onto itself, the inequality

$$card(\{x \in E \ : \ f(x) \neq x\}) < card(E)$$

holds. Further, we say that a structure S on E is almost rigid in the strong sense if, for every monomorphism $g : (E, S) \to (E, S)$ of this structure into itself, the inequality

$$card(\{x \in E \ : \ g(x) \neq x\}) < card(E)$$

holds. Let us remark that if a basic set E is finite, then the notion of an almost rigid structure (on E) may rather frequently coincide with the notion of an almost rigid structure in the strong sense. For instance, suppose that Σ is the type of an algebraic structure or the type of a structure of a linearly ordered set and let S be a structure of the type Σ defined on a finite basic set E. Then it can easily be checked that S is almost rigid if and only if S is almost rigid in the strong sense.

Of course, any rigid structure is almost rigid, but the converse assertion is not true. It is clear that any almost rigid structure in the strong sense is almost rigid, too. The following simple example shows us that there exists a graph structure which is almost rigid in the strong sense but is not rigid.

Example 1. Let x and y be any two distinct elements which do not belong to the countable set of integers $\{1, 2, ..., n, ...\}$. Let us put

$$E = \{x, y\} \cup \{1, 2, ..., n, ...\}$$

and let us define the graph structure S on the set E by the following edges:

$$\{1, x\}, \{1, y\}, \{1, 2\}, \{2, 3\}, ..., \{n, n + 1\}, ...$$

Then it is not difficult to check that S is not a rigid structure. At the same time S is almost rigid in the strong sense. Moreover, here every monomorphism of the structure S into itself moves at most two elements of the basic set E.

Let us notice, in connection with Example 1, that a structure of an infinite well ordered set without the last element gives us a simple example of a rigid structure which is not almost rigid in the strong sense.

Let E be an infinite basic set and let S be a structure of the type Σ defined on E. The following two questions naturally arise:

Question 1. Does there exist a subset X of the set E with $card(X) = card(E)$ such that the structure S_X induced on X is rigid?

Question 2. Does there exist a subset X of the set E with $card(X) = card(E)$ such that the structure S_X induced on X is almost rigid (or is almost rigid in the strong sense)?

Notice that Question 1 was extensively investigated by several authors for a topological structure and Question 2 was extensively investigated for a measurable space structure (see, e.g., the article of Shortt [9] and references given in this article).

Let us remark also, in connection with the first question, that in the most of interesting and important situations the answer to this question is negative. In particular, one of such situations is described in the next simple example.

Example 2. Let us consider the type Σ of a structure of a measurable space with the additional axiom which says that all one-element subsets of a basic set are measurable. This type of a structure can often be met in various domains of mathematics, especially in modern analysis and probability theory. Now, let E be an infinite basic set and let S be a structure of the type Σ on E. Then it is not difficult to see that, for the pair (E, S), the answer to Question 1 is negative.

Another simple example of such a situation can be obtained if we consider the type Σ of a structure of a complete graph defined on an infinite basic set.

Thus, we see that it is more perspective to investigate Question 2 concerning the existence of almost rigid (respectively, almost rigid in the strong sense) substructures induced by the original structure S. We want to notice, in connection with Question 2, that it is possible to establish some general conditions sufficient for the affirmative solution of the above-mentioned question (see, for instance, Proposition 1 below). Notice also that those general conditions are formulated in terms of partial isomorphisms or in terms of partial monomorphisms of the given structure S (let us recall that e.g. a partial monomorphism is any injective homomorphism of the form $f : (Y, S_Y) \to (E, S)$, where Y is a subset of E and S_Y is the structure on Y induced by the original structure S). Actually, we can say that one of those sufficient conditions represents an abstract version of the purely topological Lavrentiev's theorem about extensions of homeomorphisms of subsets of Polish spaces to homeomorphisms of G_δ-subsets of such spaces. This classical theorem with its various generalizations and applications is thoroughly considered in the well known monograph of Kuratowski [2].

In order to formulate Proposition 1 we need a simple auxiliary notion concerning partial homomorphisms. Namely, let

$$f : (Y, S_Y) \to (E, S), \quad g : (Z, S_Z) \to (E, S)$$

be any two partial homomorphisms. We shall say that the partial homomorphism f majorates the partial homomorphism g if f is an extension of g.

Proposition 1. *Let S be a structure on an infinite basic set E and suppose that, for each subset D of E with $card(D) = card(E)$, there exists a structure S_D on D induced by S. Suppose also that there exists a family Φ of partial monomorphisms (acting from subsets of E into E) satisfying the following two conditions:*

1) $card(\Phi) \leq card(E)$;

2) for every partial monomorphism $g : (Z, S_Z) \to (E, S)$, there is a partial monomorphism $f \in \Phi$ such that f majorates g.

Then there exists a subset X of E satisfying the next two relations:

a) $card(X) = card(E)$;

b) for an arbitrary monomorphism $h\_ : (X, S_X) \to (X, S_X)$, the *cardinality of the set*

$$\{x \in X \ : \ h(x) \neq x\}$$

is strictly less than the cardinality of X.

Consequently, the structure S_X *is almost rigid in the strong sense (in particular, this structure is almost rigid).*

Proof. Let α be the least ordinal number of cardinality $card(E)$. Obviously, we can represent the family Φ in the form

$$\Phi = \{f_\xi \ : \ \xi < \alpha\}.$$

Let us remark that the identity transformation of E belongs to the family Φ and, without loss of generality, we may assume that f_0 coincides with this transformation. Now, let us define, applying the method of transfinite recursion, an injective family

$$\{x_\xi \ : \ \xi < \alpha\}$$

of elements of the basic set E. Suppose that $\beta < \alpha$ and a partial family $\{x_\xi \ : \ \xi < \beta\}$ of elements of E has already been defined. Let us consider two sets

$$A = \{f_\xi(x_\zeta) \ : \ \xi < \beta, \ \zeta < \beta\},$$

$$B = \{f_\xi^{-1}(x_\zeta) \ : \ \xi < \beta, \ \zeta < \beta\}.$$

Evidently, we have the inequalities

$$card(A \cup B) \leq 2(card(\beta))^2 < card(E).$$

Consequently, the relation

$$E \setminus (A \cup B) \neq \emptyset$$

is true. Let x_β be an element of the set $E \setminus (A \cup B)$.

In such a way we are able to construct the required family $\{x_\xi \ : \ \xi < \alpha\}$ of elements of E. Now, let us put

$$X = \{x_\xi \ : \ \xi < \alpha\}.$$

Clearly, $card(X) = card(E)$. From the assumptions of the proposition it follows that there exists a structure S_X on X induced by the original structure S. Take an arbitrary monomorphism

$$h : (X, S_X) \to (X, S_X).$$

This monomorphism can be considered as a partial monomorphism

$$g : (X, S_X) \to (E, S).$$

According to condition 2), there exists a partial monomorphism $f \in \Phi$ such that f majorates g. Obviously, for some ordinal number $\xi < \alpha$, we have $f = f_\xi$. Taking into account the construction of the set X, it is not difficult to check that the inequality

$$card(\{x \in X \ : \ f_\xi(x) \neq x\}) < card(X)$$

is fulfilled. We also can write

$$\{x \in X \ : \ f(x) \neq x\} = \{x \in X \ : \ h(x) \neq x\}.$$

Hence, we obtain the inequality

$$card(\{x \in X \ : \ h(x) \neq x\}) < card(X),$$

which shows us that the structure S_X is almost rigid in the strong sense. Thus the proof of Proposition 1 is complete.

Let (E, S) be again a set equipped with a structure of the type Σ. We say that a mapping of the form

$$f : (Y, S_Y) \to (Z, S_Z)$$

is a partial isomorphism (acting from (E, S) into (E, S)) if Y and Z are some subsets of E, S_Y and S_Z are the structures on these subsets induced by S, and f is an isomorphism of the structure S_Y onto the structure S_Z.

The next proposition is analogous to Proposition 1.

Proposition 2. *Let S be a structure on an infinite basic set E and suppose that, for each subset D of E with $card(D) = card(E)$,* *there exists a structure S_D on D induced by S. Suppose also that there exists a family Φ of partial isomorphisms (acting from (E, S) into (E, S)) satisfying the following two conditions:*

1) $card(\Phi) \leq card(E)$;

2) for every partial isomorphism g acting from (E, S) into (E, S), there is a partial isomorphism $f \in \Phi$ such that f majorates g.

Then there exists a subset X of E satisfying the following relations:

a) $card(X) = card(E)$;

b) for an arbitrary isomorphism $h : (X, S_X) \to (X, S_X)$ the cardinality of the set

$$\{x \in X \; : \; h(x) \neq x\}$$

is strictly less than the cardinality of X.

In particular, the structure S_X is almost rigid.

Notice that the proof of Proposition 2 is quite similar to the proof of Proposition 1.

Remark 1. The result of Proposition 1 sometimes can be generalized to the case of partial morphisms which are not necessarily monomorphisms. For instance, a direct analogue of Proposition 1 can be true for those partial morphisms which have small preimages (in the sense of cardinality) of the one-element subsets of a basic set E. More precisely, if the cardinality of the set E is regular and all partial morphisms $f \in \Phi$ satisfy the inequality

$$card(f^{-1}(x)) < card(E),$$

for each element $x \in E$, then the analogue of Proposition 1 is true for such partial morphisms, too.

Remark 2. The assumption that, for each subset D of E with $card(D) = card(E)$, there exists a structure S_D on D induced by S is rather essential in the formulation of Proposition 1. This can be shown by simple examples of algebraic structures. Indeed, let us consider the set E of all integers equipped with a natural group operation - addition of numbers. It is not difficult to check that in such a case there exists a family Φ of partial monomorphisms satisfying conditions 1) and 2) of Proposition 1. But there does not exist an infinite subgroup X of

E satisfying relations a) and b) of the same proposition. Moreover, in this case all infinite subgroups of the group E are isomorphic to E and the group structure of E is not an almost rigid structure in our sense.

Remark 3. Let S be a structure on an infinite basic set E. Suppose that, for every subset D of E, there exists a structure S_D on D induced by S. Suppose also that there is a family K of subsets of E satisfying the following conditions:

1) $card(K) \leq card(E)$;

2) for each set Z belonging to K, the cardinality of the family of all monomorphisms from Z into E is less or equal to $card(E)$;

3) for any partial monomorphism

$$g : Y \to E \quad (Y \subseteq E)$$

there exists a partial monomorphism

$$f : Z \to E \quad (Z \subseteq E)$$

such that $Z \in K$ and f majorates g.

Then it is easy to see that there exists a family Φ of partial monomorphisms satisfying the assumptions of Proposition 1. Consequently, we can assert the existence of a subset X of the basic set E such that $card(X) = card(E)$ and the structure S_X on X induced by S is almost rigid in the strong sense.

Let us notice that the family K mentioned above is, as usual, an inner object for the given structure S, i.e. an inner term for S, according to the terminology of Bourbaki (see [1]). We want to notice also that condition 3) may be considered as an abstract version of Lavrentiev's theorem on extensions of homeomorphisms.

Remark 4. Let E be an infinite basic set, let S be a structure on E and let Φ be a family of partial morphisms from (E, S) into (E, S) satisfying the subsequent two conditions:

1) $card(\Phi) \leq card(E)$;

2) for every partial morphism g from (E, S) into (E, S), there is a partial morphism $f \in \Phi$ such that f majorates g.

Then there exists a mapping

$$h : E \to E$$

having the following property: for every set $X \subseteq E$ with $card(X) = card(E)$, the restriction of h to X is not a morphism from X into E.

The proof of this fact is analogous to the proof of Proposition 1. Indeed, applying the method of transfinite recursion we can construct a mapping $h : E \to E$ so that the inequality

$$card(\{x \in E : h(x) = f(x)\}) < card(E)$$

will be true for each partial morphism $f \in \Phi$.

Actually, this construction is due to Sierpiński. More precisely, Sierpiński applied the construction presented above in a particular situation where Σ is the type of a topological structure and the class of morphisms is the class of all continuous mappings. The corresponding result (due to Sierpiński and Zygmund) is formulated as follows: there exists a function

$$h : \mathbf{R} \to \mathbf{R}$$

such that its restriction to any subset X of $\mathbf{R}$ with $card(X) = card(\mathbf{R})$ is not a continuous mapping (here $\mathbf{R}$ denotes the set of all real numbers equipped with the standard order topology).

Notice that an analogous result is also true if we take the class of all Borel mappings as a class of morphisms.

Remark 5. Let E be an infinite basic set, let J be an ideal of subsets of E and let S be a structure on E.

We say that the structure S is J-rigid if, for every automorphism

$$f : (E,S) \to (E,S)$$

of this structure onto itself, we have $\{x \in E : f(x) \neq x\} \in J$.

We say that the structure S is J-rigid in the strong sense if, for every monomorphism

$$g : (E,S) \to (E,S)$$

of this structure into itself, we have $\{x \in E : g(x) \neq x\} \in J$.

Obviously, the concept of a J-rigid structure and the concept of a J-rigid structure in the strong sense are generalizations of the concepts of a rigid structure, an almost rigid structure and an almost rigid structure in the strong sense. Also, it is easy to see that some generalizations of Propositions 1 and 2 can be formulated and proved for J-rigid (J-rigid in the strong sense) mathematical structures. Moreover, if J satisfies some natural conditions, then a set X can be taken so that $X \notin J$.

Let us return to almost rigid structures and to Question 2 posed at the beginning of the paper. Namely, we wish to discuss here a natural application of Propositions 1 and 2 to the situation where the type Σ coincides with the type of a structure of a Dedekind complete linearly ordered set with some additional properties. In this situation we take the class of all increasing mappings as a class of morphisms for our type Σ. Hence, in this case the class of all monomorphisms is the class of all strictly increasing mappings.

A detailed information on linearly ordered sets (and, in particular, on Dedekind complete linearly ordered sets) can be found in the well known monograph of Sierpiński [7].

First let us consider a situation where we do not have infinite substructures almost rigid in the strong sense. Indeed, let Σ be the type of a structure of an infinite well ordered set. Obviously, Σ is simultaneously the type of a structure of an infinite, Dedekind complete, linearly ordered set. Let (E, S) be an arbitrary infinite set equipped with a structure of the type Σ. One can easily verify that there exists a monomorphism

$$f : (E, S) \to (E, S)$$

such that the equality

$$card(\{x \in E : f(x) \neq x\}) = card(E)$$

is fulfilled. Similarly, for any infinite subset X of E, there exists a monomorphism

$$g : (X, S_X) \to (X, S_X)$$

such that

$$card(\{x \in X : g(x) \neq x\}) = card(X).$$

Consequently, the structure S_X induced on the set X is not almost rigid in the strong sense.

However, we shall see below that some additional assumptions about the type Σ of a structure of an infinite, Dedekind complete, linearly ordered set imply the existence of an infinite substructure almost rigid in the strong sense.

Our further consideration needs two simple auxiliary assertions concerning linearly ordered sets.

Lemma 1. *Let $(E, \leq)$ be a Dedekind complete dense linearly ordered set and let X be a subset of E dense in E (i.e. every nonempty open subinterval of E intersects X). Then for each increasing mapping $g : X \to E$ there exists an increasing mapping $g^* : E \to E$ extending g. Moreover, if the original mapping g is strictly increasing, then the mapping g^* is strictly increasing, too.*

This lemma is well known and its proof is not difficult. Actually, the required extension g^* can be directly defined by the formula

$$g^*(e) = sup\{g(x) : x \in X \text{ and } x \leq e\},$$

where e is an arbitrary element of the basic set E. Taking into account the fact that $(E, \leq)$ is a dense linearly ordered set, we see that if g is a strictly increasing mapping, then g^* is a strictly increasing mapping, too. We also want to remark that, in general, g^* is not the unique extension of g.

Lemma 2. *Let $(E, \leq)$ be a Dedekind complete dense linearly ordered set. If the basic set E contains at least two distinct elements, then $card(E) \geq \mathbf{c}$, where $\mathbf{c}$ denotes the cardinality of the continuum.*

This lemma is well known and can easily be proved by the standard method using a dyadic system of closed bounded subintervals of E.

Let $(E, \leq)$ be an ordered set. We say that this set is isodyne if the cardinality of each nonempty open subinterval of E is equal to the cardinality of the basic set E. In other words, $(E, \leq)$ is isodyne if and only if the space E is isodyne with respect to the order topology. For example, the real line $\mathbf{R}$ is an isodyne linearly ordered set.

Let us denote by the symbol $Mon(E, E)$ the set of all strictly increasing mappings from the ordered set E into itself.

Lemma 3. *Let $(E, \leq)$ be an infinite isodyne ordered set, let γ be the least ordinal number corresponding to the cardinality of the basic set E and let*

$$Mon(E, E) = \{g_\alpha : \alpha < \gamma\}.$$

Then there exists a subset

$$X = \{x_\alpha : \alpha < \gamma\}$$

of E satisfying the following relations:

(1) the family $\{x_\alpha : \alpha < \gamma\}$ is injective; in particular, $card(X) = card(E)$;

(2) X is dense everywhere in E;

(3) for each ordinal $\alpha < \gamma$ and for any two ordinals $\beta < \alpha$, $\theta < \alpha$, we have $x_\alpha \neq g_\beta(x_\theta)$ and $x_\alpha \neq g_\beta^{-1}(x_\theta)$.

Proof. The argument is very similar to the proof of Proposition 1. Namely, we shall construct, by the method of transfinite recursion, an injective γ-sequence of points

$$\{x_\alpha : \alpha < \gamma\} \qquad (x_\alpha \in E).$$

For this purpose denote by $\{V_\alpha : \alpha < \gamma\}$ the family of all nonempty open subintervals of E and let $\{g_\alpha : \alpha < \gamma\}$ be the family of all monomorphisms from E into E. Of course, without loss of generality, we can assume that g_0 is the identity transformation of the set E. Suppose now that, for an ordinal $\alpha < \gamma$, the partial α-sequence $\{x_\beta : \beta < \alpha\}$ has already been constructed. Let us define two sets:

$$A = \{g_\beta(x_\theta) : \beta < \alpha, \theta < \alpha\},$$

$$B = \{g_\beta^{-1}(x_\theta) : \beta < \alpha, \theta < \alpha\}.$$

Obviously, the cardinality of the set $A \cup B$ is strictly less than the cardinality of the set E. Since E is isodyne, there exists an element x belonging to the set

$$V_\alpha \setminus (A \cup B).$$

Let us put $x_\alpha = x$. Therefore, using the method of transfinite recursion, we are able to construct a certain γ–sequence of elements of E. It is clear that this sequence is injective, and if we put

$$X = \{x_\alpha : \alpha < \gamma\},$$

then it is not difficult to check that the set X is a required one. Slightly changing the above argument we can prove that the required set X satisfies also the following relation:

(4) $card(X \cap V) = card(E)$, for each nonempty open subinterval V of E.

Of course, relation (4) is much stronger than relation (2). This ends the proof.

Now, we can formulate one of many results dealing with the existence of almost rigid substructures of the original mathematical structure. Here we restrict our consideration to the theory of Dedekind complete dense linearly ordered sets. The classical example of such a set is the real line **R** with its natural ordering. Another standard example is the so called Suslin line (see Example 4 below).

Proposition 3. *Let $(E, \leq)$ be an infinite dense isodyne Dedekind complete linearly ordered set and let*

$$card(Mon(E, E)) \leq card(E).$$

In other words, we can write

$$Mon(E, E) = \{g_\alpha : \alpha < \gamma\},$$

where γ is the least ordinal number corresponding to the cardinality of the basic set E. Let X be a subset of E satisfying relations (1), (2) and (3) of Lemma 3. Then the structure $(X, \leq)$ is almost rigid in the strong sense.

Proof. Let g be any monomorphism from X into X. By Lemma 1, there exists a monomorphism g^* which acts from E into E and extends g. Taking into account the definition of the set X, we have

$$card(\{x \in X : g^*(x) \neq x\}) < card(X).$$

Consequently, we also have

$$card(\{x \in X : g(x) \neq x\}) < card(X),$$

and the structure $(X, \leq)$ is almost rigid in the strong sense.

Example 3. Let us put $E = \mathbf{R}$ and let us take as $\leq$ the usual ordering of $\mathbf{R}$. Then it is easy to see that Proposition 3 can directly be applied in this case. Hence, there exists an everywhere dense subset X of $\mathbf{R}$ such that $card(X)$ is equal to the cardinality of the continuum and every strictly increasing mapping, acting from X into X, is almost identity transformation of X. We can also assume that, for each nonempty open subinterval V of $\mathbf{R}$, the intersection $X \cap V$ has the cardinality of the continuum. Moreover, we can even assume that X is a Bernstein subset of $\mathbf{R}$ (for the definition of a Bernstein subset of the real line and for the properties of such subsets, see [2], [3], [4] or [5]).

We also can consider a more general situation. Namely, let κ be an infinite cardinal number such that, for every cardinal $\lambda < \kappa$, we have the inequality

$$2^{\lambda} \leq \kappa.$$

Then there are dense isodyne Dedekind complete linearly ordered sets $(E, \leq)$ satisfying the following conditions:

1) $card(E) = 2^{\kappa}$;

2) E contains a dense subset D with $card(D) = \kappa$.

For various examples of $(E, \leq)$ with the above-mentioned properties, see e.g. the monograph of Sierpiński [7].

Consequently, for such $(E, \leq)$ we have the inequality

$$card(Mon(E, E)) \leq card(E).$$

Thus, we may apply directly Proposition 3 to $(E, \leq)$. Applying this proposition we obtain that there exists a subset X of E such that

a) $card(X) = card(E)$;

b) X is dense everywhere in E;

c) X is isodyne with respect to the induced order;

d) X is almost rigid in the strong sense with respect to the induced order.

Example 4. Let us recall that a Suslin line is a nonempty Dedekind complete dense linearly ordered set $(E, \leq)$, without the first and the last elements, satisfying the Suslin condition (i.e. the countable chain condition which says that every disjoint family of nonempty open subintervals of E is at most countable) and nonseparable in its order topology. It is well known that the existence of a Suslin line is consistent with the usual axiomatic set theory **ZFC** and is not provable from this theory (see, for instance, [6]). Let us consider briefly the question about the cardinality of a Suslin line E. On one hand, by Lemma 2, we have the inequality $card(E) \geq \mathbf{c}$. On the other hand, we have the inequality $card(E) \leq \mathbf{c}$. The latter fact can directly be deduced from each of the following two well known results:

1) the Erdös-Rado theorem of the combinatorial set theory;

2) the Arhangelskii theorem about the cardinality of a compact topological space satisfying the first countability axiom.

Notice also that, by a classical result of D.Kurepa, any Suslin line E contains an everywhere dense subset whose cardinality is equal to the first uncountable cardinal number ω_1 (the above-mentioned inequality $card(E) \leq \mathbf{c}$ follows immediately from this result). Thus, we conclude that the equality

$$card(E) = \mathbf{c}$$

holds, and we can deduce that any Suslin line E is an isodyne linearly ordered set.

Let us remark that R.Jensen showed, assuming the Axiom of Constructibility, the existence of a rigid Suslin line E (the mentioned axiom with its various consequences and applications is discussed in detail, e.g., in [6]). Furthermore, V.I.Fukson proved in [8] that if the Axiom of Constructibility holds, then there exists a Suslin line E such that, for any continuous mapping

$$f : E \to E,$$

at least one of the following two assertions is true:

a) f is a constant mapping;

b) f is the identity transformation of E.

Another interesting example (in **ZFC**) of a Dedekind complete dense isodyne linearly ordered set is the so called long line of Alexandrov. This line is a nonseparable one-dimensional connected manifold containing an everywhere dense subset of cardinality ω_1.

Example 5. Let ω denote the first infinite cardinal number. It is obvious that if the Continuum Hypothesis holds, then we have

$$\mathbf{c} = 2^{\omega} < 2^{\omega_1}.$$

The Second Continuum Hypothesis is the following set-theoretical assertion:

$$2^{\omega} = 2^{\omega_1} \qquad (SCH).$$

This assertion was considered, many years ago, by N.Luzin who also expected that it is consistent with the usual axioms of Set Theory, likely as the classical Continuum Hypothesis. Indeed, much later a number of models of Set Theory were constructed in which the Second Continuum Hypothesis holds (see, for instance, [6]). In particular, there are models of Set Theory in which we have the following equalities:

$$2^{\omega} = 2^{\omega_1} = \omega_2.$$

Actually, if we start with an arbitrary countable transitive model of **ZFC**, satisfying the Generalized Continuum Hypothesis, and apply the Cohen forcing to it, then we obtain a model of **ZFC** in which the above-mentioned equalities are fulfilled (for details, see [6]).

Assume now that the Second Continuum Hypothesis holds.

Let $(E, \leq)$ be an arbitrary Dedekind complete dense isodyne linearly ordered set containing an everywhere dense subset of cardinality ω_1. Then we have

$$card(E) = 2^{\omega_1} = 2^{\omega} = \mathbf{c}.$$

Also, it is not difficult to verify that

$$card(Mon(E, E)) \leq 2^{\omega_1} = 2^{\omega} = \mathbf{c}.$$

Therefore, in this situation we can apply Proposition 3 again and we conclude that, in theory (ZFC) & (SCH), each linearly ordered set

$(E, \leq)$ with the properties formulated above contains an everywhere dense subset X satisfying the following relations:

a) $card(X) = card(E)$;

b) X is almost rigid in the strong sense (with respect to the induced order).

In addition, we see that the required subset X of E can be constructed so that, for an arbitrary nonempty open subinterval V of E, we have the equality

$$card(X \cap V) = card(E).$$

Moreover, we can even assume that X is a Bernstein type subset of E, i.e.

$$card(X \cap P) = card((E \setminus X) \cap P) = card(E),$$

for every nonempty perfect subset P of E.

Example 6. The preceding example can be generalized to some situations where we have a Dedekind complete dense isodyne linearly ordered set $(E, \leq)$ with

$$card(E) > \mathbf{c}.$$

More precisely, let κ and λ be any two infinite cardinal numbers satisfying the equality

$$2^{\lambda} = \kappa.$$

Further, let $(E, \leq)$ be a Dedekind complete dense isodyne linearly ordered set satisfying the next two conditions:

1) $card(E) = \kappa$;

2) E contains a dense subset D with $card(D) = \lambda$.

Then there exists a dense subset X of E such that

a) $card(X) = card(E)$;

b) X is almost rigid in the strong sense (with respect to the induced order).

Moreover, we may assume that, for any nonempty open subinterval V of the set E, the equality

$$card(V \cap X) = card(E)$$

holds; in particular, X is an isodyne linearly ordered set.

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O PRAWIE SZTYWNYCH STRUKTURACH MATEMATYCZNYCH

W pracy rozważa się prawie sztywne struktury matematyczne. Zostały zbadane pewne własności takich struktur.

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