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ON VITALI CONSTRUCTION FOR COMMUTATIVE GROUPS WITH QUASIINVARIANT MEASURES

We discuss an analogue of the classical Vitali construction of a Lebesgue nonmeasurable set for uncountable commutative groups equipped with nonzero σ-finite quasiinvariant measures.

Let R be the additive group of the real line, Q be the subgroup of R consisting of all rational numbers and let λ denote the standard Lebesgue measure on R. The classical Vitali theorem (see [13]) states that every selector of the family R/Q is nonmeasurable with respect to λ (the same result holds for any countable dense subgroup H of R and for all selectors of the family R/H). This important theorem was generalized in several directions (see, e.g., [1], [2], [5], [8], [11], [12], [14]). In particular, some generalizations and analogues of the Vitali theorem were obtained for locally compact topological groups equipped with the Haar measure (see, for instance, [4] or [3], Section 4). The aim of the present paper is to show that many questions arising naturally in connection with this theorem can completely be solved, for uncountable commutative groups, without the aid of any topological methods. Namely, we shall establish, in our further considerations, the corresponding analogue of the Vitali theorem for uncountable commutative groups equipped with nonzero σ-finite quasiinvariant measures.

The notation and terminology used in the present paper are standard. As usual, ω denotes the first infinite cardinal number, ω_1 denotes the first uncountable cardinal number, 2^ω denotes the cardinality of the continuum. A set Y is countable if $card(Y) \leq \omega$. If f is an arbitrary function, then $dom(f)$ denotes the domain of f. Let $\{X_i : i \in I\}$ be a family of nonempty pairwise disjoint sets and let $E = \cup\{X_i : i \in I\}$. We say that a set $X \subseteq E$ is a partial selector of $\{X_i : i \in I\}$ if, for each index $i \in I$, we have the inequality $card(X \cap X_i) \leq 1$. Furthermore, we say that a set $X \subseteq E$ is a selector of $\{X_i : i \in I\}$ if, for each $i \in I$, we have the equality $card(X \cap X_i) = 1$. Obviously, every partial selector of $\{X_i : i \in I\}$ can be extended to a selector of $\{X_i : i \in I\}$. Various selectors appear naturally in the situation where two groups G and H are given such that $H \subseteq G$. For instance, let $(G, +)$ be a commutative group and let H be a subgroup of G. We say that a set $X \subseteq G$ is an H-selector (respectively, a partial H-selector) if X is a selector (respectively, a partial selector) of the family G/H canonically associated with H. In our considerations we suppose, as a rule, that H is a nontrivial (i.e. nonzero) subgroup of G.

Let $(G, \cdot)$ be an arbitrary group and let μ be a measure defined on some σ-algebra of subsets of G. We recall that μ is a G-invariant measure if $dom(\mu)$ is invariant with respect to the group of all left translations of G and $\mu(gY) = \mu(Y)$ for each $g \in G$ and for each $Y \in dom(\mu)$. A more general concept is the concept of a G-quasiinvariant measure. For a given measure μ on G, let us denote by $Z(\mu)$ the class of all μ-measure zero sets. We recall that μ is a G-quasiinvariant measure if the classes of sets $dom(\mu)$ and $Z(\mu)$ are invariant with respect to the group of all left translations of G. Clearly, every G-invariant measure is simultaneously a G-quasiinvariant measure. The converse assertion is not true, in general. Let us observe that if we have a nonzero σ-finite G-quasiinvariant (G-invariant) measure μ on G, then we can easily define a probability G-quasiinvariant measure ν on G such that $dom(\nu) = dom(\mu)$ and $Z(\nu) = Z(\mu)$ (in other words, μ and ν are equivalent measures). This simple observation will be applied below.

Of course, one can introduce the concept of a G-invariant measure (G-quasiinvariant measure) with respect to the group of all right translations of G. If the original group G is commutative, then two concepts of a G-invariant (G-quasiinvariant) measure are identical.

Now, let us fix an uncountable commutative group $(G,+)$ with a nonzero σ-finite G-quasiinvariant measure μ defined on some σ-algebra of subsets of G. In connection with the above–mentioned Vitali theorem, the following four questions arise in a natural way.

Question 1. Let H be an infinite countable subgroup of G. Is it true that all H-selectors are nonmeasurable with respect to μ?

Question 2. Let H be a nontrivial countable subgroup of G. Is it true that there exists at least one H-selector nonmeasurable with respect to μ?

Question 3. Let H be a countable subgroup of G and let $G/H = \{X_i : i \in I\}$. Is it true that there exists a subset J of I such that all selectors of the partial family $\{X_i : i \in J\}$ are nonmeasurable with respect to μ?

Question 4. Let H be an uncountable subgroup of G such that $card(G/H) = card(G)$. Is it true that there exists at least one H-selector nonmeasurable with respect to μ?

One can show that the answer to Question 1 is negative. Moreover, in [6] a measure μ is constructed satisfying the following conditions:

1) μ is defined on some σ-algebra of subsets of the real line R;

2) μ is a nonzero nonatomic σ-finite measure;

3) $dom(\lambda)$ is contained in $dom(\mu)$;

4) for each Lebesgue measurable subset X of R with $\lambda(X) = 0$, we have $\mu(X) = 0$;

5) for each Lebesgue measurable subset X of R with $\lambda(X) > 0$, we have $\mu(X) = +\infty$;

6) μ is invariant with respect to the group of all isometric transformations of R (in particular, μ is invariant with respect to the group of all translations of R);

7) there exists a μ-measurable Q-selector.

Thus, we see that there exists a Vitali subset of R measurable with respect to a certain nonzero σ-finite R-invariant measure on R.

Also it is not difficult to show that the answer to Question 4 is negative (see, e.g., [7]). Indeed, let us put $G = R^2$, where R^2 is the

Euclidean plane, and let us take as H the subgroup $\{0\} \times R$ of G. Evidently, we have the equalities

$$card(G/H) = card(R) = card(G) = 2^{\omega}.$$

Denote by λ^2 the usual two-dimensional Lebesgue measure on G. Further, let P be the σ-ideal of subsets of G, generated by the family of all H-selectors. It can easily be checked that P possesses the following properties:

a) P is invariant under the group of all translations of G;

b) $\lambda^2_*(X) = 0$, for each set X belonging to P, where λ^2_* denotes the inner measure associated with λ^2.

Starting with these two properties of P and applying the standard argument (see, e.g., [7]), it is not difficult to prove the existence of a measure μ on G such that

(1) μ is an extension of λ^2;

(2) μ is a G-invariant measure;

(3) P is contained in $dom(\mu)$;

(4) $\mu(X) = 0$ for each set $X \in P$.

In particular, we see that all H-selectors are measurable with respect to μ.

Remark 1. Let $n > 0$ be a natural number, R^n be the n-dimensional Euclidean space and let λ^n denote the n-dimensional Lebesgue measure on R^n. Finally, let H be an arbitrary uncountable subgroup of the additive group of R^n. Suppose that Martin's Axiom and the negation of the Continuum Hypothesis are fulfilled. Then it can be proved (see [1]) that there exists a measure μ on R^n satisfying the following conditions:

(1) μ is an extension of λ^n;

(2) μ is an R^n-invariant measure;

(3) all H-selectors are measurable with respect to μ;

(4) if X is an arbitrary H-selector, then $\mu(X) = 0$.

Now, we are going to show that the answers to Questions 2 and 3 are positive. Moreover, we shall establish a much stronger result (see Theorem 1 below). First we shall formulate several auxiliary propositions.

Lemma 1. *Let $\{X_i : i \in I\}$ be a family of pairwise disjoint sets such that $card(X_i) > 1$, for all indices $i \in I$, and let X be a partial selector of $\{X_i : i \in I\}$. Then there exist two selectors Y_1 and Y_2 of $\{X_i : i \in I\}$ satisfying the equality $Y_1 \cap Y_2 = X$.*

This lemma is trivial. From it we immediately obtain the next proposition.

Lemma 2. *Let E be a set equipped with a measure μ and let $\{X_i : i \in I\}$ be a partition of E such that $card(X_i) > 1$, for all $i \in I$. Suppose also that there exists a partial selector of $\{X_i : i \in I\}$ nonmeasurable with respect to the measure μ. Then there exists a selector of $\{X_i : i \in I\}$ nonmeasurable with respect to μ.*

Let G_1 be a group equipped with a probability G_1-quasiinvariant measure μ_1, let G_2 be another group and let f be an arbitrary homomorphism from G_1 onto G_2. We denote

$$S = \{Y \subseteq G_2 : f^{-1}(Y) \in dom(\mu_1)\}.$$

Obviously, S is a σ-algebra of subsets of the group G_2, invariant with respect to the group of all left translations of G_2. We define a functional μ_2 on S by the formula

$$\mu_2(Y) = \mu_1(f^{-1}(Y)) \qquad (Y \in S).$$

It is easy to see that the following proposition holds.

Lemma 3. *μ_2 is a probability G_2-quasiinvariant measure on G_2. Moreover, if the original measure μ_1 is G_1-invariant, then μ_2 is G_2-invariant.*

Notice, in connection with Lemma 3, that if μ_1 is an arbitrary σ-finite G_1-quasiinvariant (respectively, G_1-invariant) measure on the group G_1, then the measure μ_2 on the group G_2, defined by the same formula, is G_2-quasiinvariant (respectively, G_2-invariant) but we cannot assert, in general, that μ_2 is σ-finite.

The next lemma plays the key role in our further considerations.

Lemma 4. *Let G be an uncountable commutative group equipped with a nonzero σ-finite G-quasiinvariant measure μ. Then there exists a subgroup Γ of G nonmeasurable with respect to μ.*

Lemma 4 was proved in [9]. Here we want to remark only that the proof of this lemma is essentially based on some combinatorial properties of the Ulam $(\omega \times \omega_1)$-matrix and on a well known theorem from group theory, concerning the algebraic structure of commutative groups (more precisely, the above-mentioned theorem states that every commutative group can be represented as the union of a countable family of subgroups each of which is the direct sum of cyclic groups).

In addition, we may assume in Lemma 4, without loss of generality, that the subgroup Γ of G is uncountable. Indeed, it is sufficient to apply Lemma 4 to any G-quasiinvariant extension ν of μ such that $dom(\nu)$ contains all countable subsets of G.

From Lemma 4 we can deduce the following

Lemma 5. *Let G be a commutative group with a nonzero σ-finite G-quasiinvariant measure μ and let H be a subgroup of G satisfying the inequality $card(G/H) > \omega$. Then there exists a subgroup Γ of G such that*

1) H is contained in Γ;

2) Γ is nonmeasurable with respect to μ.

Proof. We may assume, without loss of generality, that μ is a probability measure on G. Let us denote by f the canonical homomorphism from the given group G onto the factor group G/H and let us put

$$S = \{Y \subseteq G/H : f^{-1}(Y) \in dom(\mu)\}.$$

Further, let us define a measure ν on the σ-algebra S by the formula

$$\nu(Y) = \mu(f^{-1}(Y)) \qquad (Y \in S).$$

According to Lemma 3, ν is a probability (G/H)-quasiinvariant measure on the uncountable group G/H. According to Lemma 4, there exists a subgroup Γ^* of G/H nonmeasurable with respect to ν. Let us put $\Gamma = f^{-1}(\Gamma^*)$. Then one can easily verify that Γ is a subgroup of G nonmeasurable with respect to μ and

$$H = ker(f) = f^{-1}(0) \subseteq f^{-1}(\Gamma^*) = \Gamma.$$

Thus, Lemma 5 is proved.

Lemma 6. *Let G be an uncountable group equipped with a nonzero σ-finite G-quasiinvariant measure μ. Then there exists a subset of G nonmeasurable with respect to μ.*

This lemma was proved in [5]. Notice that a stronger result can be established for σ-finite invariant measures (see [11]). But the method used in [11] does not work for σ-finite quasiinvariant measures.

Now, we can formulate the following result.

Theorem 1. *Let G be an uncountable commutative group equipped with a nonzero σ-finite G-quasiinvariant measure μ and let H be a countable subgroup of G. Denote by $G/H = \{X_i : i \in I\}$ the partition of G canonically associated with H. Then there exists a subset J of I such that*

1) the union of the partial family $\{X_i : i \in J\}$ is a subgroup of G nonmeasurable with respect to μ;

2) all selectors of $\{X_i : i \in J\}$ are nonmeasurable with respect to μ;

3) if H is a nontrivial subgroup of G, then there exists an H-selector nonmeasurable with respect to μ.

Proof. Applying Lemma 5, we see that there exists a subgroup Γ of G such that

$$H \subseteq \Gamma, \quad \Gamma \notin dom(\mu).$$

Since $\Gamma/H \subseteq G/H$, we can write

$$\Gamma/H = \{X_i : i \in J\} \subseteq \{X_i : i \in I\},$$

for some $J \subseteq I$. Obviously, we have the equality

$$\Gamma = \cup\{X_i : i \in J\}.$$

Consequently, relation 1) holds for $\{X_i : i \in J\}$. Further, let X be an arbitrary selector of $\{X_i : i \in J\}$. We assert that X is nonmeasurable with respect to μ. Suppose otherwise, i.e. $X \in dom(\mu)$. Then we have

$$\Gamma = \cup\{h + X : h \in H\},$$

where all sets $h + X$ are μ-measurable. Taking into account the fact that H is a countable subgroup of G, we get $\Gamma \in dom(\mu)$ which yields a contradiction. Thus, X does not belong to $dom(\mu)$, and relation 2) holds for $\{X_i : i \in J\}$. Finally, applying Lemma 2 to the partition G/H of G, we immediately obtain that relation 2) implies relation 3). The proof of Theorem 1 is complete.

Remark 2. Unfortunately, Theorem 1 cannot be generalized to the class of all uncountable groups with nonzero σ-finite quasiinvariant measures. Indeed, Shelah proved in [10] that there exists a group G with the following properties:

a) $card(G) = \omega_1$;

b) G does not contain a proper uncountable subgroup.

Let us take such a group G and let us fix a countable subgroup H of G. Further, denote by S the σ-algebra of subsets of the group G, generated by the family of all countable subsets of G. One can easily define a probability G-invariant measure μ on S such that $\mu(Y) = 0$ for each countable subset Y of G. Now, it is clear that, for (G, μ) and H, an analogue of Theorem 1 does not hold.

However, we have the following result (cf. [7]).

Theorem 2. *Let G be an arbitrary uncountable group equipped with a nonzero σ-finite G-quasiinvariant measure μ and let $\{X_i : i \in I\}$ be a partition of G such that*

$$1 < card(X_i) \leq \omega,$$

for all indices $i \in I$. Then there exists at least one selector of $\{X_i : i \in I\}$ nonmeasurable with respect to the measure μ. In particular, if H is a nontrivial countable subgroup of G and $\{X_i : i \in I\}$ is an injective family of all left (right) H-orbits in G, then there exists a selector of $\{X_i : i \in I\}$ nonmeasurable with respect to μ.

Proof. According to Lemma 6, there is a subset Y of G nonmeasurable with respect to μ. Taking into account the inequalities $card(X_i) \leq \omega$ for all $i \in I$, we easily deduce that the set Y can be represented in the form

$$Y = \cup\{Y_n : n < \omega\},$$

where each set Y_n is a partial selector of $\{X_i : i \in I\}$. Since Y does not belong to $dom(\mu)$, there exists a natural number n such that Y_n also does not belong to $dom(\mu)$. Finally, applying Lemma 2, we conclude that there exists at least one selector of $\{X_i : i \in I\}$ nonmeasurable with respect to μ.

Remark 3. Let E be a set equipped with a measure μ and let $\{X_i : i \in I\}$ be a partition of E such that $1 < card(X_i) \leq \omega$ for all indices $i \in I$. Actually, the argument used in the proof of Theorem 2 shows that the next two assertions are equivalent:

1) there exists a subset of E nonmeasurable with respect to μ;

2) there exists a selector of $\{X_i : i \in I\}$ nonmeasurable with respect to μ.

We can prove some analogues of the preceding results in a more general situation. Namely, let G be an uncountable group, S be a σ-algebra of subsets of G and let P be a σ-ideal of subsets of G such that $P \subset S$. Suppose also that the following relations are fulfilled:

a) S is invariant under the group of all left translations of G;

b) P is invariant under the group of all left translations of G;

c) the pair (S, P) satisfies the Suslin condition (i.e. the countable chain condition).

Then a result similar to Theorem 2 holds for G, (S, P) and a non-trivial countable subgroup H of G. In addition, if G is a commutative group, then a result similar to Theorem 1 holds for G, (S, P) and a countable subgroup H of G. The proofs of those results are based on the corresponding analogues of lemmas presented above.

In particular, we can formulate the following topological result.

Theorem 3. *Let G be an uncountable commutative group and let T be a topology on G such that*

a) (G, T) is a second category topological space;

b) the σ-algebra of sets having the Baire property in (G, T) is invariant under the group of all translations of G;

c) the σ-ideal of first category sets in (G, T) is invariant under the group of all translations of G;

d) the space (G, T) satisfies the Suslin condition (i.e. the countable chain condition).

Further, let H be a countable subgroup of G and let $G/H = \{X_i : i \in I\}$ be the partition of G canonically associated with H. Then there exists a subset J of I such that

1) the union of the partial family $\{X_i : i \in J\}$ is a subgroup of G without the Baire property in (G,T);

2) all selectors of $\{X_i : i \in J\}$ do not have the Baire property in (G,T);

3) if H is a nontrivial subgroup of G, then there exists an H-selector without the Baire property in (G,T).

In the similar way we can formulate a topological result analogous to Theorem 2.

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KONSTRUKCJA VITALIEGO DLA PRZEMIENNYCH GRUP Z MIARAMI PRAWIE NIEZMIENNICZYMI

W pracy rozważa się analogon klasycznej konstrukcji Vitaliego dla nieprzeliczalnych grup przemiennych z σ–skończonymi miarami prawie niezmienniczymi.

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