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Chiral fermions, massless particles and Poincare covariance

Krzysztof Andrzejewski, Agnieszka Kijanka-Dec, Piotr Kosiński *, Paweł Maślanka

Department of Computer Science, Faculty of Physics and Applied Informatics, University of Lodz, Pomorska 149/153, 90-236 Lodz, Poland

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ABSTRACT

The coadjoint orbit method is applied to the construction of Hamiltonian dynamics of massless particles of arbitrary helicity. The unusual transformation properties of canonical variables are interpreted in terms of nonlinear realizations of Poincare group. The action principle is formulated in terms of new space–time variables with standard transformation properties.

1. Introduction

Recently, triangle anomalies, chiral fermions and Berry curvature in momentum space, their interrelations and role played in various physical phenomena have attracted much attention [1–23]. Much of the research consists in exploring anomaly-related phenomena in kinetic theory. An important point here is that, assuming weak external fields and weak particle interactions, one can rely to large extent on (semi)classical approximation. For example, instead of using the Weyl equation one can describe massless chiral fermions of helicity $\frac{1}{2}$ by the action functional

$$S=\int\left(\left(\vec{p}+e\vec{A}\right)\cdot\dot{\vec{x}}-\left(|\vec{p}|+e\Phi\right)-\vec{\alpha}\cdot\dot{\vec{p}}\right)dt \tag{1}$$

involving the vector potential $\vec{\alpha}(\vec{p})$ describing the Berry monopole in momentum space. Eq. (1) can be derived from Weyl Hamiltonian by considering semiclassical approximation to the path-integral representation of a transition amplitude [16] or, alternatively, using wave-packet approach [24].

The main problem with Eq. (1) is that it lacks manifest Lorentz symmetry even in the absence of external fields. This is the more surprising that it has been derived from explicitly covariant Weyl theory. To shed some light on the problem the authors of Ref. [16] proposed a modified transformation law for particle dynamical variables which is consistent in the sense that it leaves the dynamics following from the action (1) invariant and reduces to the standard Lorentz symmetry if the additional terms which arise due to the nonzero helicity are neglected. However, their proposal is exotic in the sense that: (i) it contains additional, helicity-dependent, terms mentioned above; (ii) the group composition rule closes only “on-shell”.

A deep analysis of the resulting situation has been given in the nice recent papers [14,22,23]. In particular, Duval et al. not only extended the results of Ref. [16] to the case of full Poincare symmetry but they reconsidered the whole problem in more general framework provided by the Souriau sympletic approach to dynamics [25]. They were able to derive the Poincare symmetry for chiral fermions by showing that the latter can be obtained from Souriau’s model of relativistic massless spinning particle by the procedure called “spin enslaving”.

Let us note that some of the apparently paradoxical features of Lorentz transformation laws for particles with nonzero spin (massive case) or helicity (massless case) appear to be unavoidable consequences of the group structure and basic conservation laws. It has been noticed long time ago [26] that the generators of Poincare symmetry for massless particle of nonzero helicity cannot be constructed out of canonical variables obeying standard canonical commutation rules and having standard transformation properties; if it were possible, the helicity would acquire more than one value within irreducible representation of Poincare group. Another nice argument in favor of “exotic” transformation has been given in Ref. [16] (see also [15]) where the zero impact parameter collision of two massless particles of nonvanishing helicities was considered. By applying the Lorentz boost along the direction of motion of one incoming particle it is shown there that such a boost must result in “side jump” in order to fulfill the angular momentum conservation law. Similar side jumps which depend only on the kinematics of the problem appear, for example, in impurity scattering caused by spin-orbit interaction [27]. This phenomenon seems also to have its counterpart in optics in the form of the relativistic Hall effect of light [28–34,15] (see also [35–37]).

* Corresponding author.
E-mail address: pkosinsk@uni.lodz.pl (P. Kosiński).

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The reason for the existence of the above described specific side jumps can be also traced back to the question of defining the center of mass of relativistic extended spinning body [38,39,14].

In the present paper, inspired by Ref. [22], we study further the invariance properties of the action functional (1). Our starting point is the construction of the Hamiltonian dynamics for massless particles with arbitrary helicity. The main tool we use is the coadjoint orbits method [25,40–42]; it has been already applied to the dynamics of relativistic particles in a number of references [25,43–49]. We classify the orbits corresponding to massless particles of the given helicities and construct the generators of Poincare group in terms of canonical variables. We find explicit form of the stability subgroup of a "canonical" point on the orbit and reinterpret the whole construction in terms of nonlinear realizations of Poincare group. This allows for quite natural interpretation of "exotic" transformation properties of coordinate variables. It is shown that the action principle can be put in the form which does not depend on the value of helicity; the latter enters only the transformation properties of basic variables. On the other hand, if one insists on having standard transformation properties of basic space–time variables, the action functional becomes helicity-dependent and exhibits the gauge symmetry, the gauge group being the stability subgroup mentioned above. The initial description is then obtained by an appropriate gauge fixing which is not covariant under the action of full Poincare group. Therefore, the action of the latter on initial variables is a composition of left action by standard space–time transformations supplemented by a gauge transformation. This provides alternative way of looking at the unusual transformation properties of dynamical variables representing massless particles.

2. Classical massless particles

We adopt the convention $g_{\mu\nu} = diag(+---)$. The light-cone coordinates are defined by $x^{\pm} = \frac{1}{\sqrt{2}}(x^0 \pm x^3)$. Let k be fixed but arbitrary parameter having the momentum dimension and let $k^\mu = (k,0,0,k)$ be the standard null vector. Denote by $L_k \subset SO(3,1)$ the stability subgroup of k^μ. Any element Λ of $SO(3,1)$ can be decomposed as follows

$$\Lambda = B \cdot D \cdot R, \quad D \in L_k, \; R \in L_k \tag{2}$$

where in the light-cone basis (x^+, x^-, x^1, x^2) the matrices B, D and R take the form

$$B = \begin{pmatrix} {\Lambda^+}_+ & 0 & 0 & 0 \\ {\Lambda^-}_+ & \frac{1}{{\Lambda^+}_+} & \frac{{\Lambda^1}_+}{{\Lambda^+}_+} & \frac{{\Lambda^2}_+}{{\Lambda^+}_+} \\ {\Lambda^1}_+ & 0 & 1 & 0 \\ {\Lambda^2}_+ & 0 & 0 & 1 \end{pmatrix} \tag{3}$$

$$D = \begin{pmatrix} 1 & \frac{{\Lambda^+}_-}{{\Lambda^+}_+} & d_1 & d_2 \\ 0 & 1 & 0 & 0 \\ 0 & d_1 & 1 & 0 \\ 0 & d_2 & 0 & 1 \end{pmatrix} \tag{4}$$

$$R = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\alpha & \sin\alpha \\ 0 & 0 & -\sin\alpha & \cos\alpha \end{pmatrix} \tag{5}$$

and

$$d_{1,2} \equiv \frac{{\Lambda^{1,2}}_- {\Lambda^+}_+ - {\Lambda^{1,2}}_+ {\Lambda^+}_-}{{\Lambda^+}_+} \tag{6}$$

$$\cos\alpha \equiv \frac{{\Lambda^1}_1 {\Lambda^+}_+ - {\Lambda^1}_+ {\Lambda^+}_1}{{\Lambda^+}_+} \tag{7}$$

$$\sin\alpha \equiv \frac{{\Lambda^1}_2 {\Lambda^+}_+ - {\Lambda^1}_+ {\Lambda^+}_2}{{\Lambda^+}_+} \tag{8}$$

The decomposition (2) is singular at some points because the principal bundle $(SO(3,1), L_k)$ is nontrivial but this fact does not affect the reasoning. Note that B parametrize the coset manifold $SO(3,1)/L_k$. Denote by (Λ, a) the elements of Poincare group $\mathcal{P}$, the composition law being $(\Lambda, a) \cdot (\Lambda', a') = (\Lambda\Lambda', \Lambda a' + a)$. An infinitesimal element $g = (I + \omega, \epsilon)$ can be written as

$$g = I + i\epsilon^\mu P_\mu - \frac{i}{2}\omega^{\mu\nu} M_{\mu\nu} \tag{9}$$

with P_μ and $M_{\mu\nu} = -M_{\nu\mu}$ being the generators for translations and Lorentz transformations, respectively. Denote by ζ_μ and $\zeta_{\mu\nu} = -\zeta_{\nu\mu}$ the coordinates in the dual space to Lie algebra of $\mathcal{P}$. The coadjoint action of $\mathcal{P}$ reads

$$Ad^*_{(\Lambda,a)}\zeta_\mu = {\Lambda_\mu}^\nu \zeta_\nu \tag{10}$$

$$Ad^*_{(\Lambda,a)}\zeta_{\mu\nu} = {\Lambda_\mu}^\alpha {\Lambda_\nu}^\beta \zeta_{\alpha\beta} + (a_\mu {\Lambda_\nu}^\alpha - a_\nu {\Lambda_\mu}^\alpha)\zeta_\alpha \tag{11}$$

The dual space is equipped with invariant Poisson structure which can be read off from the basic commutation rules of Poincare algebra:

$$\{\zeta_\mu, \zeta_\nu\} = 0 \tag{12}$$

$$\{\zeta_{\mu\nu}, \zeta_\alpha\} = g_{\nu\alpha}\zeta_\mu - g_{\mu\alpha}\zeta_\nu \tag{13}$$

$$\{\zeta_{\mu\nu}, \zeta_{\alpha\beta}\} = g_{\mu\beta}\zeta_{\nu\alpha} + g_{\nu\alpha}\zeta_{\mu\beta} - g_{\mu\alpha}\zeta_{\nu\beta} - g_{\nu\beta}\zeta_{\mu\alpha} \tag{14}$$

The coadjoint orbits are classified by selecting the values of the invariants corresponding to the Casimir operators

$$\mathcal{M}^2 \equiv \zeta^\mu \zeta_\mu \tag{15}$$

$$\mathcal{W}^2 \equiv w^\mu w_\mu, \quad w^\mu = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}\zeta_\nu \zeta_{\alpha\beta} \tag{16}$$

Note the following Poisson brackets following from Eqs. (12)–(16):

$$\{w^\mu, \zeta_{\rho\sigma}\} = \delta^\mu_\rho w_\sigma - \delta^\mu_\sigma w_\rho \tag{17}$$

$$\{w^\mu, w^\nu\} = \epsilon^{\mu\nu\rho\sigma}\zeta_\rho w_\sigma \tag{18}$$

We are interested in coadjoint orbits corresponding to $\mathcal{M}^2 = 0$, $\mathcal{W}^2 = 0$ and $\zeta^0 > 0$. Due to the former condition any such orbit contains a point $(\underline{\zeta}_\mu, \underline{\zeta}_{\mu\nu})$ with $\underline{\zeta}_\mu = (k,0,0,-k) \equiv k_\mu$. Once $\underline{\zeta}_\mu$ is fixed, $\mathcal{W}^2 = 0$ yields

$$\underline{\zeta}_{01} - \underline{\zeta}_{31} = 0 \tag{19}$$

$$\underline{\zeta}_{02} - \underline{\zeta}_{32} = 0 \tag{20}$$

Consider now the action of the subgroup of $\mathcal{P}$ consisting of elements (h, a), where $h \in L_k$. Using Eqs. (10), (11), (19) and (20) we easily conclude that to any orbit under consideration there belongs the "canonical" point:

$$\underline{\zeta}_\mu = k_\mu$$

$$\underline{\zeta}_{\mu\nu} = \begin{cases} 0, & (\mu\nu) \neq (12), (21) \\ -s, & (\mu\nu) = (12) \\ s, & (\mu\nu) = (21) \end{cases} \tag{21}$$

Note that at this point

$$\underline{w}^\mu = sk^\mu \tag{22}$$

Both sides of Eq. (22) are fourvectors under the coadjoint action of Poincare group; therefore, anticipating slightly the notation,

$$w^\mu = sp^\mu \tag{23}$$

We see that the orbit is uniquely characterized by the single parameter s which describes helicity.

Let $\mathcal{P}_k \subset \mathcal{P}$ be the stability subgroup of canonical point (21) under the coadjoint action of Poincare group. It consists of the elements $(h, a(h))$ where $h = DR \in L_k$ and $a(h)$ is defined by

$$a^0 = a^3\text{-arbitrary}, \qquad a^1 = \frac{s}{\sqrt{2}k} d_2, \qquad a^2 = \frac{-s}{\sqrt{2}k} d_1 \tag{24}$$

with $d_{1,2}$ being defined by Eq. (4).

The Lie algebra of $\mathcal{P}_k$ is spanned by

$$E_1 \equiv M_{01} + M_{31} + \frac{s}{\sqrt{2}k} P_2 \tag{25}$$

$$E_2 \equiv M_{02} - M_{23} - \frac{s}{\sqrt{2}k} P_1 \tag{26}$$

$$P_+ = \frac{1}{\sqrt{2}}(P_0 + P_3) \tag{27}$$

$$J = M_{12} \tag{28}$$

The relevant commutation rules read

$$[J, E_i] = i\epsilon_{ik} E_k \tag{29}$$

$$[E_i, E_k] = \frac{2i \cdot s}{k} \epsilon_{ik} P_+ \tag{30}$$

$$[P_+, \cdot] = 0 \tag{31}$$

$\mathcal{P}_k$ is, therefore, the centrally extended $E(2)$ group.[1] L_k is its image under the canonical homomorphism $\mathcal{P} \to SO(3,1)$. The orbit under consideration is isomorphic to the coset manifold:

$$V = \mathcal{P}/\mathcal{P}_k \tag{32}$$

In order to parametrize V let us consider an arbitrary element $(\Lambda, a) \in \mathcal{P}$. First, we decompose Λ according to Eq. (2)

$$\Lambda = B \cdot (DR) \equiv B \cdot h \tag{33}$$

Let $y^\mu = (0, \vec{y})$; consider the decomposition

$$(\Lambda, a) = (B, y) \cdot (h, a(h)) \tag{34}$$

where $a(h)$ is given by Eq. (24). Eqs. (33) and (34) imply

$$a^\mu = (Ba(h))^\mu + y^\mu \tag{35}$$

Eqs. (33) and (35) can be solved to yield h, $a(h)$, B and y (the solution is unique but somewhere singular due to the nontriviality of the relevant bundle, as mentioned above). The pair (B, y) parametrizes the coset manifold V and, consequently, the coadjoint orbit

$$(\zeta_\mu, \zeta_{\mu\nu}) = Ad^*_{(B,y)}(\underline{\zeta}_\mu, \underline{\zeta}_{\mu\nu}) \tag{36}$$

with $\underline{\zeta}_\mu$, $\underline{\zeta}_{\mu\nu}$ given by Eq. (21). Finally, we parametrize B as follows:

$$p^\mu = \Lambda^\mu{}_+ k^+ = \Lambda^\mu{}_+ \cdot \sqrt{2} \cdot k, \qquad p^\mu p_\mu = 0 \tag{37}$$

Using Eqs. (36) and (37) one easily finds

$$\zeta_\mu = p_\mu \tag{38}$$

$$\zeta_{12} = y_1 p_2 - y_2 p_1 \tag{39}$$

$$\zeta_{23} = y_2 p_3 - y_3 p_2 + \frac{sp_1}{\sqrt{2}p^+} \tag{40}$$

$$\zeta_{31} = y_3 p_1 - y_1 p_3 + \frac{sp_2}{\sqrt{2}p^+} \tag{41}$$

$$\zeta_{01} = -y_1 p_0 + \frac{sp_2}{\sqrt{2}p^+} \tag{42}$$

$$\zeta_{02} = -y_2 p_0 - \frac{sp_1}{\sqrt{2}p^+} \tag{43}$$

$$\zeta_{03} = -y_3 p_0 \tag{44}$$

We see that the classical massless particles define the nonlinear realization of the Poincare group corresponding to the stability subgroup $\mathcal{P}_k$. The variables $\vec{y}$ and $\vec{p}$ provide independent coordinates on coadjoint orbit/coset manifold (they are Goldstone or preferred variables in terminology of Ref. [50]). Their transformation properties are derived either from coadjoint action of Poincare group on ζ_μ and $\zeta_{\mu\nu}$ or from its left action on the coset manifold $\mathcal{P}/\mathcal{P}_k$. The relevant transformation rules can be described as follows. The momentum variables p_μ transform separately. Translations act trivially on them while the action of infinitesimal Lorentz transformations $\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu$ reads

$$\delta p_k = \beta_k |\vec{p}| + \epsilon_{klm} \omega_l p_m \tag{45}$$

where $\beta_k \equiv \omega_{k0}$, $\epsilon_{ikl}\omega_l = \omega_{ik}$. The translation subgroup is the kernel of the realization on momentum variables which is the nonlinear realization of Lorentz group determined by the $E(2)$ subgroup (stability subgroup of k^μ). In spite of the fact that the third axis plays a distinguished role the realization linearizes on rotations. The y-variables transform in a more complicated way. Consider again Lorentz transformations. They read

$$\delta\vec{y} = \vec{\omega} \times \vec{y} - (\vec{\beta} \cdot \vec{y}) \frac{\vec{p}}{|\vec{p}|} + \vec{\nabla}_p \Psi(p) \tag{46}$$

where

$$\Psi(p) = s\left(\frac{\omega_1 p_1 + \omega_2 p_2 + \beta_1 p_2 - \beta_2 p_1}{\sqrt{2}p^+} \right) \tag{47}$$

It is not difficult to check that the transformations (45)–(47) are canonical

$$\delta(\cdot) = \left\{ (\cdot), \frac{1}{2}\omega^{\mu\nu}\zeta_{\mu\nu} \right\} \tag{48}$$

As a next step we find the Poisson brackets for y's and p's. Using Eqs. (12)–(14) and (38)–(44) one computes

$$\{y_i, y_k\} = 0 \tag{49}$$

$$\{y_i, p_k\} = \delta_{ik} \tag{50}$$

$$\{p_i, p_k\} = 0 \tag{51}$$

In terms of y's rotations linearize only on the subgroup of rotations around the third axis. However, one can make things explicitly rotationally invariant by passing to the coordinates $\vec{x}$ defined as follows:

$$y_1 = x_1 + \frac{sp_2}{\sqrt{2}p^+ p^0} \tag{52}$$

$$y_2 = x_2 - \frac{sp_1}{\sqrt{2}p^+ p^0} \tag{53}$$

$$y_3 = x_3 \tag{54}$$

[1] This interesting property can be also inferred from the discussions presented in Refs. [14,22,23] and, in form of symmetry of Wess–Zumino-like action, from Ref. [58].

Then we find

$$\zeta_{0i} = -p_0 x_i \tag{55}$$

$$\zeta_{ij} = x_i p_j - x_j p_i + \frac{s\epsilon_{ijk}p_k}{p^0} \tag{56}$$

The price one has to pay for simplifying the transformation properties is that the new variables are no longer Darboux ones. In fact, new Poisson brackets read

$$\{x_i, x_j\} = -\frac{s\epsilon_{ijk}p_k}{(p^0)^3} \tag{57}$$

$$\{x_i, p_j\} = \delta_{ij} \tag{58}$$

$$\{p_i, p_j\} = 0 \tag{59}$$

In terms of new variables rotations, as it has been mentioned above, take the standard form. On the other hand, the boosts read

$$\delta x_i = -(\beta_j \cdot x_j)\frac{p_i}{|\vec{p}|} + \frac{s\epsilon_{ijk}\beta_j p_k}{|\vec{p}|^2} \tag{60}$$

$$\delta p_i = \beta_i |\vec{p}| \tag{61}$$

For completeness let us write out the action of translation subgroup (I, a). It reads

$$\delta x_i = a_i - a_0\frac{p_i}{|\vec{p}|}$$

$$\delta p_i = 0 \tag{62}$$

This agrees with the identification $H = p^0 = |\vec{p}|$.

3. Poincare symmetry

To derive the symmetry transformations we note that our symmetry is a dynamical one: the Hamiltonian belongs to the Lie algebra of symmetry group and, in general, does not Poisson-commute with other generators. Their time evolution is given by the one parameter subgroup of adjoint transformations generated by the Hamiltonian. Expressing the initial $(t = 0)$ generators in terms of actual ones yields the conserved charges which generate the symmetry. In our case the new conserved generators read

$$\tilde{\zeta}_\mu = \zeta_\mu \tag{63}$$

$$\tilde{\zeta}_{ij} = \zeta_{ij} \tag{64}$$

$$\tilde{\zeta}_{0i} = \zeta_{0i} + \zeta_i t \tag{65}$$

By virtue of Eqs. (63)–(65) we conclude that the symmetry transformations corresponding to the boosts are modified according to

$$\delta x_i = -(\beta_k x_k)\frac{p_i}{|\vec{p}|} + \beta_i t + \frac{s\epsilon_{ijk}\beta_j p_k}{|\vec{p}|^2} \tag{66}$$

The same result is obtained by applying the original transformations (60) and (61) to initial variables and propagating them to the moment t with the help of equations of motion.

Concluding, the symmetry transformations are obtained through nonlinear action of Poincare group on the coset manifold defined by the subgroup of Poincare group related to $\mathcal{P}_k$ by a time-dependent internal automorphism generated by the Hamiltonian. In other words, let $\vec{x} = \vec{x}(t, \vec{x}_0, \vec{p}_0)$, $\vec{p} = \vec{p}(t, \vec{x}_0, \vec{p}_0)$ be the solution to the equations of motion; the change of variables $(\vec{x}, \vec{p}, t) \to (\vec{x}_0, \vec{p}_0, t)$ yields the nonlinear realization with $\vec{x}_0$, $\vec{p}_0$ being the preferred variables parametrizing $\mathcal{P}/\mathcal{P}_k$ while t is the adjoint variable [50] transforming trivially under the action of $\mathcal{P}_k$.

Transformation rules (66) can be put in yet another form. Within the Hamiltonian formalism the symmetry transformations do not involve the redefinition of time. The symmetries including the change of time variable are accommodated by recomputing the values of dynamical variables back to initial time with the help of canonical equations of motion; for any dynamical variable η the relation between the Hamiltonian and Lagrangian form of symmetries reads $\delta_H \eta = \delta_L \eta - \dot{\eta}\delta t$. Keeping this in mind we rewrite the transformation rules (66) as

$$\delta t = \beta_k x_k \tag{67}$$

$$\delta x_i = \beta_i t + \frac{s\epsilon_{ijk}\beta_j p_k}{|\vec{p}|^2} \tag{68}$$

$$\delta p_i = \beta_i |\vec{p}| \tag{69}$$

For $s = 0$ one arrives at the standard Lorentz transformation rules. However, for $s \neq 0$ the above transformation rules close only "on-shell" [16]. Indeed, the relevant differential boost generators read

$$M_{0k} = i\left(x_k\frac{\partial}{\partial t} + t\frac{\partial}{\partial_x k} + \frac{s\epsilon_{klj}p_l}{|\vec{p}|^2}\frac{\partial}{\partial x_j} + |\vec{p}|\frac{\partial}{\partial p_k}\right) \tag{70}$$

The corresponding commutation rule takes the form

$$[M_{0k}, M_{0m}] = -iM_{km} - \frac{2s\epsilon_{kml}p_l}{|\vec{p}|^2}\left(\frac{\partial}{\partial t} + \frac{p_j}{|\vec{p}|}\frac{\partial}{\partial x_j}\right) \tag{71}$$

and reduces to the standard form on trajectories $x_k - \frac{p_k t}{|\vec{p}|} = const$.

4. Quantum theory

It is easy to quantize the classical theory formulated above. We start with diagonalizing the momenta yielding the momentum representation. As the momentum variables transform in a standard way it is convenient to use the explicitly invariant scalar product

$$(f, g) = \int \frac{d^3\vec{p}}{2|\vec{p}|}\overline{f(\vec{p})}g(\vec{p}) \tag{72}$$

Due to the canonical relations (49)–(51) y's are basically p-derivatives. However, we should take into account the hermiticity condition with respect to the scalar product (72). Therefore,

$$\vec{y} = \sqrt{|\vec{p}|}\left(\frac{i\partial}{\partial\vec{p}}\right)\frac{1}{\sqrt{|\vec{p}|}} = \frac{i\partial}{\partial\vec{p}} - \frac{i\vec{p}}{2|\vec{p}|^2} \tag{73}$$

Now, one can construct generators according to the equations (38)–(44); to this end one has to perform symmetrization $y_i p_0 \to \frac{1}{2}(y_i p_0 + p_0 y_i)$. The resulting generators read

$$P_\mu = p_\mu \tag{74}$$

$$M_{12} = i\left(p_2\frac{\partial}{\partial p_1} - p_1\frac{\partial}{\partial p_2}\right) - s \tag{75}$$

$$M_{23} = i\left(p_3\frac{\partial}{\partial p_2} - p_2\frac{\partial}{\partial p_3}\right) + \frac{sp_1}{p_0 - p_3} \tag{76}$$

$$M_{31} = i\left(p_1\frac{\partial}{\partial p_3} - p_3\frac{\partial}{\partial p_1}\right) + \frac{sp_2}{p_0 - p_3} \tag{77}$$

$$M_{01} = -i|\vec{p}|\frac{\partial}{\partial p_1} + \frac{sp_2}{p_0 - p_3} \tag{78}$$

$$M_{02} = -i|\vec{p}|\frac{\partial}{\partial p_2} - \frac{sp_1}{p_0 - p_3} \tag{79}$$

$$M_{03} = -i|\vec{p}|\frac{\partial}{\partial p_3} \tag{80}$$

It is easy to check that $M_{\mu\nu}$ and P_μ obey Poincare algebra. If we demand that it integrates to the representation of universal covering $ISL(2,C)$ of Poincare group s must be integer or halfinteger. The above representation coincides with that given, for example, in Ref. [51] or [52].

5. Action principle

It is well-known that the Kirillov form defining Poisson brackets on coadjoint orbit is related to the Cartan forms on the relevant coset manifold [53]. Applying the prescription given in [53] to the case of Poincare symmetry we define

$$\Omega(p,y) \equiv \Omega^\mu(p,y)\underline{\zeta}_\mu + \Omega^{\mu\nu}(p,y)\underline{\zeta}_{\mu\nu} \tag{81}$$

where (p,y) parametrize the coset manifold and

$$(p,y)^{-1}d(p,y) = i\Omega^\mu(p,y)P_\mu + i\Omega^{\mu\nu}(p,y)M_{\mu\nu} \tag{82}$$

Then

$$\tilde{\Omega} \equiv d\Omega \tag{83}$$

is the relevant Kirillov form. Explicit computation yields

$$\Omega = -p^i dy^i = -p^i dx^i + \alpha^i(\vec{p})dp^i \tag{84}$$

with

$$\vec{\alpha}(\vec{p}) = s\left(\frac{-p^2}{p^0p^+}, \frac{p^1}{p^0p^+}, 0\right) \tag{85}$$

being the vector potential of the monopole.

According to the general theory the action functional yielding correct equations of motion reads

$$S = \int \left(-\Omega - Hdt\right) \tag{86}$$

leading to

$$S = \int \left(\vec{p}\cdot\dot{\vec{y}} - |\vec{p}|\right)dt = \int \left(\vec{p}\cdot\dot{\vec{x}} - |\vec{p}| - \vec{\alpha}(\vec{p})\cdot\dot{\vec{p}}\right)dt \tag{87}$$

The first form of the action integral confirms the conclusion that $(\vec{y},\vec{p})$ are Darboux variables. Let us note that it does not depend on the helicity s. Before entering more sophisticated aspects of action principle let us make some remarks. The textbook action for massive relativistic particle reads

$$S = -m\int ds = -m\int \sqrt{1-\dot{\vec{y}}^2}dt \tag{88}$$

The $m \to 0$ limit cannot be taken directly. However, one can pass to the Hamiltonian form which is straightforward for $m \neq 0$ and yields

$$S = -\int p_\mu dy^\mu, \qquad y^0 = t, \qquad p_\mu p^\mu = 0 \tag{89}$$

It is now easy to take the limit $m \to 0$ which gives Eq. (87). In terms of y, p variables the action has the universal form as it does not depend on the helicity value. The latter enters only the transformation rule. Let us write it in "Lagrangian" form:

$$\delta y^0 = \vec{\beta}\cdot\vec{y} \tag{90}$$

$$\delta\vec{y} = \vec{\omega}\times\vec{y} + \vec{\beta}y^0 + \vec{\nabla}_p\Psi(p) \tag{91}$$

$$\delta\vec{p} = \vec{\omega}\times\vec{p} + \vec{\beta}p^0 \tag{92}$$

$$\delta p^0 = \vec{\beta}\cdot\vec{p} \tag{93}$$

$\Psi(p)$ is proportional to the helicity and this is the only term where s enters. The additional contribution to the action integrand reads

$$\begin{aligned}\vec{p}d\left(\vec{\nabla}_p\Psi(p)\right) &= d\left(\vec{p}\cdot\vec{\nabla}_p\Psi(p)\right) - d\vec{p}\cdot\vec{\nabla}_p\Psi(p) = \\ &= d\left(\vec{p}\cdot\vec{\nabla}_p\Psi(p) - \Psi(\vec{p})\right)\end{aligned} \tag{94}$$

which proves the invariance of action principle.

One can pose the question whether and how the space–time variables transforming according to the formula $x \to \Lambda x + a$ can be built into the theory. New formalism should be equivalent to the one based on coadjoint orbits. Therefore, all dynamical variables should be constructed out of group elements. Assume the global symmetry is identified with (say) left action of the group on itself. From the form of nonlinear group action we conclude that our dynamics must be invariant under the right action of stability subgroup viewed as the gauge group. In fact, the relevant dynamical variables parametrize the coset space. If one works with the variables parametrizing the whole group, those corresponding to the subgroup must be redundant and should be eliminated by a symmetry transformations. Writing schematically the nonlinear action on coset space

$$gw = w'h(w,g) \tag{95}$$

we see that in order to eliminate the subgroup variables one has to act from the right with the stability subgroup elements which are generally time-dependent; we are dealing with gauge symmetry.

It is quite easy to write out the action integral on group manifold which is invariant under the global action of this group by left multiplication and the local action of some its subgroup by right multiplication provided this subgroup is the stability group of some point on coadjoint action. Let G be a Lie group, $H \subset G$ its subgroup leaving invariant the element $\underline{\xi}_\alpha$ of dual space to Lie algebra of G. Writing the Cartan–Maurer form as

$$g^{-1}dg = i\eta^\alpha(g)A_\alpha \tag{96}$$

where A_α are the generators of G and putting [53]

$$\eta(g) \equiv \eta^\alpha(g)\underline{\xi}_\alpha \tag{97}$$

one easily finds that $\omega(g)$ is invariant under the global left action of G and, up to a total differential, under the local right action of H. Therefore, the first-order action

$$S = \int \left(-\eta(g)\right) \tag{98}$$

defines invariant dynamics of G/H (because of the gauge symmetry under the right action of H).

Let apply the above construction to the Poincare group. One has

$$(\Lambda,z)^{-1}d(\Lambda,z) = (\Lambda^{-1}d\Lambda, \Lambda^{-1}dz) \tag{99}$$

Keeping in mind the form of "canonical" point (21) we arrive easily at the following form of invariant action

$$S = -\int \left(k\left({\Lambda_\mu}^0 - {\Lambda_\mu}^3\right)dz^\mu - \frac{is}{2}Tr\left(J\Lambda^{-1}d\Lambda\right)\right) \tag{100}$$

with $J = M_{12}$. This Wess–Zumino like action was considered in Refs. [54–58]. (See also [59].) It posses the expected symmetries under:

- the global Poincare transformations:

$$(\tilde{\Lambda},a) : (\Lambda,z) \longrightarrow (\tilde{\Lambda}\Lambda, \tilde{\Lambda}z + a) \tag{101}$$

– the local $\mathcal{P}_k$ transformations:

$$\delta\Lambda = i\theta^a(t)\Lambda\tilde{E}_a + i\varphi(t)\Lambda M \tag{102}$$

$$\delta z^\mu = \frac{s}{k}\left(\theta^1(t)\Lambda^\mu{}_2 - \theta^{(2)}(t)\Lambda^\mu{}_1\right) + + a(t)(\Lambda^\mu{}_0 + \Lambda^\mu{}_3) \tag{103}$$

$$\tilde{E}_1 \equiv M_{01} + M_{31}, \qquad \tilde{E}_2 \equiv M_{02} - M_{23} \tag{104}$$

The first order action (100) exhibits gauge symmetry. A careful analysis of the emerging constraints leads, via Dirac method, to the conclusion that it describes, as expected, the massless helicity s particles (cf., for example, Ref. [58]). One can also proceed by fixing an appropriate gauge. To this end we recall the decomposition (2)–(8) together with the identification (37). It follows that one can fix the gauge such that $\Lambda = B$ with matrix elements being parametrized by fourmomentum p^μ. Moreover, the parameter function $a(t)$ can be chosen in such a way that $z^0 \equiv t$ (in other words the invariant evolution parameter can be replaced by time).

Fixing the gauge as above we arrive at the following simple action

$$S = \int (\vec{p}\cdot\dot{\vec{y}} - |\vec{p}|)dt \tag{105}$$

which coincides with Eq. (89); here $\vec{y}$ denotes the spatial part of gauge-transformed z^μ.

The findings of previous sections can be now rephrased as follows. The gauge fixing condition breaks the explicit global Poincare invariance. The Poincare transformation must be supplemented by an appropriate gauge transformation which restores the gauge. This makes the final transformation rule more complicated.

Let us now consider the coupling to the external electromagnetic field. The minimal coupling is achieved by adding the term $eA_\mu(z)\dot{z}^\mu$ to the Lagrangian. The z variables transform standardly under Poincare group so $A_\mu(z)$ have the standard meaning. The action takes the form

$$S = \int \Big(-(\Lambda_{\mu 0} + \Lambda_{\mu 3})kdz^\mu + eA_\mu(z)dz^\mu - - \frac{is}{2}Tr(J\Lambda^{-1}d\Lambda)\Big) \tag{106}$$

Note that the above action, while preserving the standard gauge invariance related to electromagnetic coupling, seems to break the gauge symmetry related to the right action of stability subgroup. This could imply that some of the gauge degrees of freedom become real dynamical variables. The related ambiguity in including the interaction has been discussed in Refs. [22,23]. The problem of interaction will be treated in more detail in the forthcoming paper [60].

Another interesting question concerns the relation between the present formalism for massless particles with nonzero helicities and the massive ones with spin based on twistor theory [61–63]. Also the relation with other models of massless particles [64–67] is worth of study.

After submitting this paper an additional somewhat related reference has been brought to our attention which we find interesting: D. Capasso, D. Sarkar, Phys. Rev. D 89 (2014) 084012.

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References

[1] D.T. Son, P. Surówka, Phys. Rev. Lett. 103 (2009) 191601.
[2] R. Loganayagam, Anomaly induced transport in arbitrary dimensions, arXiv:1106.0277.
[3] R. Loganayagam, P. Surówka, J. High Energy Phys. 1204 (2012) 097.
[4] S. Dubovsky, L. Hui, A. Nicolis, D.T. Son, Phys. Rev. D 85 (2012) 085029; S. Dubovsky, L. Hui, A. Nicolis, Phys. Rev. D 89 (2014) 045016.
[5] K. Fukushima, D.E. Kharzeev, H.J. Warringa, Phys. Rev. D 78 (2008) 074033.
[6] K. Fukushima, D.E. Kharzeev, H.J. Warringa, Phys. Rev. Lett. 104 (2010) 212001.
[7] G. Basar, G.V. Dunne, D.E. Kharzeev, Phys. Rev. Lett. 104 (2010) 232301.
[8] M.A. Stephanov, Y. Yin, Phys. Rev. Lett. 109 (2012) 162001.
[9] D.T. Son, N. Yamamoto, Phys. Rev. Lett. 109 (2012) 181602.
[10] J.-W. Chen, S. Pu, Q. Wang, X.-N. Wang, Phys. Rev. Lett. 110 (2013) 262301.
[11] D.T. Son, N. Yamamoto, Phys. Rev. D 87 (2013) 085016.
[12] M. Stone, V. Dwivedi, Phys. Rev. D 88 (2013) 045012.
[13] V. Dwivedi, M. Stone, J. Phys. A 47 (2014) 025401.
[14] M. Stone, V. Dwivedi, T. Zhou, Phys. Rev. D 91 (2015) 025004.
[15] M. Stone, V. Dwivedi, T. Zhou, Wigner translations and the observer-dependence of the position of massless spinning particles, arXiv:1501.04586.
[16] J.-Y. Chen, D.T. Son, M.A. Stephanov, H.-U. Yee, Y. Yin, Phys. Rev. Lett. 113 (2014) 182302.
[17] E. Megias, M. Valle, Second-order partition function of a non-interacting chiral fluid in 3 + 1 dimensions, arXiv:1408.0165.
[18] K.A. Sohrabi, J. High Energy Phys. 03 (2015) 014.
[19] C. Manuel, J.M. Torres-Rincon, Phys. Rev. D 90 (2014) 076007.
[20] J.-W. Chen, J.-Y. Pang, S. Pu, Q. Wang, Phys. Rev. D 89 (2014) 094003.
[21] P.-M. Zhang, P.A. Horvathy, Anomalous Hall Effect for semiclassical chiral fermions, arXiv:1409.4225.
[22] C. Duval, P.A. Horvathy, Phys. Rev. D 91 (2015) 045013.
[23] C. Duval, M. Elbistan, P.A. Horvathy, P.-M. Zhang, Phys. Lett. B 742 (2015) 322.
[24] M.-C. Chang, Q. Niu, J. Phys. Condens. Matter 20 (2008) 193202.
[25] J.-M. Souriau, Structure of Dynamical Systems: A Sympletic View of Physics, Birkhäuser, Boston, 1997.
[26] B.-S. Skagerstam, Localization of massless spinning particles and the Berry phase, arXiv:hep-th/9210054.
[27] L. Berger, Phys. Rev. B 2 (1970) 4559.
[28] K. Bliokh, Yu. Bliokh, Phys. Lett. A 333 (2004) 181.
[29] M. Onoda, S. Murakami, N. Nagaosa, Phys. Rev. Lett. 93 (2004) 083901.
[30] C. Duval, Z. Horvath, P.A. Horvathy, J. Geom. Phys. 57 (2007) 925.
[31] C. Duval, Z. Horvath, P.A. Horvathy, Phys. Rev. D 74 (2006) 021701(R).
[32] K. Bliokh, A. Niv, V. Kleiner, E. Hasman, Nat. Photonics 2 (2008) 748.
[33] K. Bliokh, J. Opt. A 11 (2009) 094009.
[34] K.Y. Bliokh, F. Nori, Phys. Rev. Lett. 108 (2012) 120403.
[35] R. Karplus, J.M. Luttinger, Phys. Rev. 95 (1954) 1154.
[36] E.N. Adams, E.I. Blount, J. Phys. Chem. Solids 10 (1959) 286.
[37] P.A. Horvathy, Phys. Lett. A 359 (2006) 705.
[38] C.W. Misner, K.S. Thorne, J.A. Wheeler, Gravitation, W.H. Freeman and Co., San Francisco, 1973.
[39] L.F. Costa, C. Herdeiro, J. Natario, M. Zilhao, Phys. Rev. D 85 (2012) 024001.
[40] A. Kirillov, Elements of the Theory of Representations, Springer, Berlin, New York, 1972.
[41] B. Kostant, Lect. Notes Math., vol. 170, Springer, Berlin, 1970.
[42] R. Giachetti, Riv. Nuovo Cimento 4 (1981) 1.
[43] C. Duval, Ann. Inst. Henri Poincaré A 25 (1976) 345.
[44] C. Duval, P.A. Horvathy, Ann. Phys. 142 (1982) 10.
[45] N.M.J. Woodhouse, Geometric Quantization, Clarendon Press, 1997.
[46] V. Nair, R. Ray, S. Roy, Phys. Rev. D 86 (2012) 025012.
[47] D. Capasso, V. Nair, J. Tekel, Phys. Rev. D 88 (2013) 085025.
[48] D. Karabali, V. Nair, Phys. Rev. D 90 (2014) 105018.
[49] G. Monteiro, A. Abanov, V. Nair, Hydrodynamics with gauge anomaly: variational principle and Hamiltonian formulation, arXiv:1410.4833.
[50] S. Coleman, J. Wess, B. Zumino, Phys. Rev. 177 (1969) 2239; C.G. Callan, S. Coleman, J. Wess, B. Zumino, Phys. Rev. 177 (1969) 2247.
[51] Yu.M. Shirokov, J. Exp. Theor. Phys. 21 (1951) 748 (in Russian).
[52] Yu.M. Shirokov, Dokl. Akad. Nauk SSSR 94 (1954) 857 (in Russian).
[53] J.E. Marsden, T. Ratiu, Introduction to Mechanics and Symmetry, Springer, 2002.
[54] A.P. Balachandran, G. Marmo, B.-S. Skagerstam, A. Stern, Gauge Theories and Fibre Bundles: Applications to Particle Dynamics, Lect. Notes Phys., vol. 188, Springer, 1983.
[55] B.-S. Skagerstam, A. Stern, Phys. Scr. 24 (1981) 493.
[56] B.-S. Skagerstam, A. Stern, Nucl. Phys. B 294 (1987) 636.
[57] A.P. Balachandran, G. Marmo, B.-S. Skagerstam, A. Stern, Phys. Lett. B 89 (1980) 199.

[58] M.V. Atre, A.P. Balachandran, T.R. Govindarajan, Int. J. Mod. Phys. A 2 (1987) 453.
[59] D. Karabali, V.P. Nair, Phys. Rev. D 90 (2014) 105018.
[60] K. Andrzejewski, et al., in progress.
[61] L. Mezincescu, A.J. Routh, P.K. Townsend, Supertwistors and massive particles, arXiv:1312.2768.
[62] S. Fedoruk, J. Lukierski, Phys. Lett. B 733 (2014) 309.
[63] J.A. de Azcarraga, S. Fedoruk, J.M. Izquierdo, J. Lukierski, Two-twistor particle models and free massive higher spin fields, arXiv:1409.7169.
[64] M. Plyushchay, Mod. Phys. Lett. A 4 (1989) 837.
[65] M. Plyushchay, Phys. Lett. B 243 (1990) 383.
[66] M. Plyushchay, Phys. Lett. B 240 (1990) 133.
[67] S. Klishevich, M. Plyushchay, Phys. Lett. B 459 (1999) 201.