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# THE COMPARISON OF DETERMINATION METHODS OF LOSS DISTRIBUTION IN CREDIT PORTFOLIO IN CREDITRISK+ MODEL

**Abstract.** CreditRisk+ is one of the portfolio methods used in credit risk management. Assumptions of the model, methods of loss determination and loss distribution function in the whole credit portfolio will be depicted in this dissertation. The first numerical method – designed by Credit Suisse First Boston in 1997 – tried to describe the model using the Panjer's recursions. At present, there are numerous different algorithms which enable determination of loss distribution function. Two of them will be described in detail and collated. The first algorithm is based on probability generation function, while the second one uses the Fourier inversion.

**Key words:** credit risk,CreditRisk+, probability generating function,inverse Fourier transform.

## I. INTRODUCTION OF CREDITRISK+ MODEL

Notations in standard CreditRisk+ Model:

$N$ – number of obligors,

$L_j$ – loss if the *j*th obligor default (for $j = 1,\ldots,N$),

$p_j$ – rating parameter which describes the probability that the obligor j will default within one year (for $j = 1,\ldots,N$),

$T$ – time horizon,

$K$ – number of sectors,

$a_j^k$ – the affiliation of *j*th obligor to *k*th sector ( for $j = 1,\ldots,N$ and $k = 1,\ldots,K$ ),

$a_j^0$ – idiosyncratic risk (for $j = 1,\ldots,N$ ).

These affiliations have to fulfill these conditions:

$$\forall_{\substack{j=1,\ldots,N \\ k=0,\ldots,K}} a_j^k \geq 0 \text{ and } \forall_{j=1,\ldots,N} \sum_{k=0}^{K} a_j^k = 1 . \tag{1}$$

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Random variables $R_1,...,R_K$ describe changes in sectors of economy and they are given by independent Gamma distribution with parameters $\left(\frac{1}{\sigma_k^2},\frac{1}{\sigma_k^2}\right)$.

Let $Y_j$ be random variable, which describes moment of *j*th obligor default. We assume that $Y_1,\ldots,Y_N$ are independent exponential random variables with intensity $\lambda_j$ for $j=1,\ldots,N,$ where

$$\lambda_j := p_j \cdot (a_j^0 + \sum_{k=1}^{K} a_j^k \cdot R_k)\,. \tag{2}$$

Now, we can define random variable $X:\Omega \to \mathbf{R}$ which describes loss of all credit portfolio as:

$$X := \sum_{j=1}^{N} I_j \cdot L_j \,, \tag{3}$$

where

$I_j$ – the default indicator of the *j*th obligor:

$$I_j = \begin{cases} 1, & \text{gdy } Y_j \le T \\ 0, & \text{p.p.} \end{cases} \tag{4}$$

The characteristic function $\varphi_X(s)$ is given by[1]

$$\varphi_X(s) = \exp\left\{\sum_{j=1}^{N} a_j^0 p_j T(e^{iL_j s}-1) - \sum_{k=1}^{K}\frac{1}{\sigma_k^2}\ln\left(1+\sigma_k^2 T\sum_{j=1}^{N} a_j^k p_j(1-e^{iL_j s})\right)\right\}. \tag{5}$$

Knowing this function, it is possible to obtain cumulant generating function as:

$$C_X(t) = \sum_{j=1}^{N} a_j^0 p_j T(e^{L_j t}-1) - \sum_{k=1}^{K}\frac{1}{\sigma_k^2}\ln\left(1+\sigma_k^2 T\sum_{j=1}^{N} a_j^k p_j(1-e^{L_j t})\right). \tag{6}$$

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[1] O. Reiss, *Fourier inversion algorithms for Generalized CreditRisk+ models and an extension to incorporate market risk*, WIAS-Preprint No. 817, 2003.

Because first cumulant $\kappa_1$ describes mean, and second cumulant $\kappa_2$ is equivalent to variance, it is possible to obtain both mean and variance of credit loss random variable. So the first two moments of $X$ are given by:

$$\mathrm{E}[X] = \kappa_1 = C'_X(0) = T\sum_{j=1}^{N} a_j^0 p_j L_j + T\sum_{k=1}^{K}\sum_{j=1}^{N} a_j^k p_j L_j = T\sum_{j=1}^{N} p_j L_j \,, \tag{7}$$

$$Var[X] = \kappa_2 = C''_X(0) = T\sum_{j=1}^{N} a_j^0 p_j (L_j)^2 + T\sum_{k=1}^{K}\sum_{j=1}^{N} a_j^k p_j (L_j)^2 + T^2\sum_{k=1}^{K}\sigma_k^2\left(\sum_{j=1}^{N} a_j^k p_j L_j\right)^2$$

$$= T\sum_{j=1}^{N} p_j (L_j)^2 + T^2\sum_{k=1}^{K}\sigma_k^2\left(\sum_{j=1}^{N} a_j^k p_j L_j\right)^2 \,, \tag{8}$$

$$\mathrm{E}[X^2] = Var[X] + \left(\mathrm{E}[X]\right)^2 = T\sum_{j=1}^{N} p_j (L_j)^2 + T^2\left(\sum_{j=1}^{N} p_j L_j\right)^2 + T^2\sum_{k=1}^{K}\sigma_k^2\left(\sum_{j=1}^{N} a_j^k p_j L_j\right)^2 \,. \tag{9}$$

## II. THE COMPARSION OF DETERMINATION METHODS OF LOSS DISTRIBUTION

First document, which was presented by Credit Suisse First Boston in 1997, based on Panjer's recursions. Nevertheless, after that publication, a lot of scientist started giving some thought to other methods – more efficient which in the much faster way would lead for achieving the interesting result. Below two expressed algorithms will stay. First algorithm, which was thought by G. Giese, based on probability generating function. Second is relying on the statement using the opposite Fourier transform. Both algorithms will be described in details and analysed.

Algorithm using probability generating function

So that it is possible to obtain the probability generating function, one should remember that a random variable, for which we appoint it, must assume natural values. Therefore every loss will be treated as the multiple of the individual amount of loss and it will be described as follows:

$$L_j = l_j v \,, \tag{10}$$

where:

$v$ – amount of loss for one individual loss unit,

$l_j$ – number of loss units ($l_j = 1,2,...$ ), Then the random variable $X$ assumes values which are a total multiple and therefore it is possible to appoint the probability generating function for this random variable $G_{\frac{X}{v}}$.

Using the relation:

$$G_{\frac{X}{v}}(z) = \varphi_{\frac{X}{v}}(-i \ln z) = \varphi_X\left(-\tfrac{i}{v} \ln z\right) \tag{11}$$

and formula (6) we receive:

$$G_{\frac{X}{v}}(z) = \exp\left\{\sum_{j=1}^{N} a_j^0 p_j T(z^{l_j} - 1) - \sum_{k=1}^{K} \frac{1}{\sigma_k^2} \ln\left(1 + \sigma_k^2 T \sum_{j=1}^{N} a_j^k p_j (1 - z^{l_j})\right)\right\}. \tag{12}$$

The probability generating function may be written in the form of the series expansion of $G_{\frac{X}{v}}$:

$$G_{\frac{X}{v}}(z) = \sum_{n=0}^{\infty} \mathrm{P}\left(\frac{X}{v} = n\right) z^n = \sum_{n=0}^{\infty} \mathrm{P}(X = nv) z^n . \tag{13}$$

The idea of this algorithm is to find the series expansion of $G_{\frac{X}{v}}$ at $z_0 = 0$.

To obtain the credit loss of the portfolio, using the following lemma will be essential.

**Lemma 1.**

*Let $\varepsilon > 0$. Let consider two series expansion*

$$f(z) := \sum_{n=0}^{\infty} a_n z^n \tag{14}$$

*and*

$$g(z) := \sum_{n=0}^{\infty} b_n z^n, \tag{15}$$

*which are convergent for* $|z| < \varepsilon$. *Let*

$$\underset{|z|<\varepsilon}{\forall} f(z) > 0. \tag{16}$$

*If moreover*

$$\underset{|z|<\varepsilon}{\forall} f(z) = \exp(g(z)), \tag{17}$$

*then*

$$a_0 = \exp(b_0) \tag{18}$$

*and*

$$a_n = \sum_{k=1}^{n} \frac{k}{n} a_{n-k} b_k \ \ for \ \ n = 1,2,... \tag{19}$$

*An inverse relation is given by:*

$$b_0 = \ln(a_0) \tag{20}$$

*and*

$$b_n = \frac{1}{a_0}\left( a_n - \sum_{k=1}^{n-1} \frac{k}{n} a_{n-k} b_k \right). \tag{21}$$

**Proof**

Let assumptions of the lemma will be fulfilled.

Let notice, that $f(0) = a_0$ and $a_0 > 0$. Dealing by analogy with the function $g$ we receive, that $g(0) = b_0$.

Using (18), we receive

$$a_0 = f(0) = \exp\big(g(0)\big) = \exp(b_0), \tag{22}$$

which is equivalent the formula that $b_0 = \ln a_0$.

Let noticed now, that formula (17) may be written as follows:

$$\underset{|z|<\varepsilon}{\forall} \quad g(z) = \ln\big(f(z)\big). \tag{23}$$

Determining derivative of function $g$ which is described by (23) and (15), we receive:

$$\sum_{n=0}^{\infty} n b_n z^{n-1} = \frac{1}{\sum_{n=0}^{\infty} a_n z^n} \sum_{n=0}^{\infty} n a_n z^{n-1} \quad \Leftrightarrow \quad \sum_{n=0}^{\infty} n b_n z^{n-1} \cdot \sum_{n=0}^{\infty} a_n z^n = \sum_{n=0}^{\infty} n a_n z^{n-1} \tag{24}$$

Using now definition of product of series expansion in the Cauchy meaning, we have:

$$\sum_{n=0}^{\infty} \sum_{k=0}^{n} k b_k z^{k-1} a_{n-k} z^{n-k} = \sum_{n=0}^{\infty} n a_n z^{n-1} \tag{25}$$

Thus

$$\sum_{n=0}^{\infty} z^{n-1} \sum_{k=0}^{n} k b_k a_{n-k} = \sum_{n=0}^{\infty} n a_n z^{n-1} . \tag{26}$$

From theorem about the evenness of series expansion, we receive:

$$\sum_{k=0}^{n} k b_k a_{n-k} = n a_n \tag{27}$$

$$a_n = \sum_{k=0}^{n} \frac{k}{n} b_k a_{n-k} = \sum_{k=1}^{n} \frac{k}{n} b_k a_{n-k} . \tag{28}$$

and so an equality (19) is real.

Transforming again the equality (27) in the following form

$$\sum_{k=1}^{n-1} k b_k a_{n-k} + n b_n a_0 = n a_n \tag{29}$$

we receive that

$$b_n = \frac{1}{a_0} \left( a_n - \sum_{k=1}^{n-1} \frac{k}{n} a_{n-k} b_k \right). \tag{30}$$

□

Let the $M$ indicate the largest possible loss from the credit portfolio measured in individuals loss units $l_j$, that $M = \sum_{j=1}^{N} l_j$ .

It is simultaneously number of rates for appointing in the formula (13).

Let accept the following markers:

$$G_{\frac{X}{v}}(z) = \exp \underbrace{\left\{ \sum_{j=1}^{N} a_j^0 p_j T (z^{l_j} - 1) a - \sum_{k=1}^{K} \frac{1}{\sigma_k^2} \underbrace{\ln \underbrace{\left( 1 + \sigma_k^2 T \sum_{j=1}^{N} a_j^k p_j (1 - z^{l_j}) \right)}_{A_k(z)}}_{B_k(z)} \right\}}_{D(z)} \tag{31}$$

Let start from $A_k(z)$. Let notice that rate standing before $z^0$ :

$$ {}_k a_0 = 1 + \sigma_k^2 \sum_{j=1}^{N} a_j^k p_j T \tag{32} $$

is bigger or equal 1. Since $l_j \geq 1$, so all remaining rates

$$ {}_k a_n = - \sum_{\{j:\, l_j = n\}} \sigma_k^2 a_j^k p_j T \tag{33} $$

are smaller or equal 0.

Let notice that $A_k(z) > 0$ and $B_k(z) := \ln(A_k)$. Using lemma 1, it is possible to obtain dependences between rates of function $A_k$ and $B_k$, as follows:

$$ {}_k b_0 = \ln\left({}_k a_0\right), \tag{34} $$

$$ {}_k b_n = \frac{1}{{}_k a_0} \left( {}_k a_n - \sum_{j=1}^{n-1} \frac{j}{n} \cdot {}_k b_j \cdot {}_k a_{n-j} \right). \tag{35} $$

Because ${}_k a_0 \geq 1$ and ${}_k a_n < 0$ for $n > 0$, so ${}_k b_0 \geq 0$. Let notice also, that

$$ {}_k b_1 = \frac{1}{{}_k a_0} \cdot {}_k a_1 = \frac{{}_k a_1}{{}_k a_0}. \tag{36} $$

Since ${}_k a_0 \geq 1$ and ${}_k a_1 < 0$, therefore ${}_k b_1 < 0$. By analogy ${}_k b_2 < 0$, ${}_k b_3 < 0$, and so far. In the end, we receive that ${}_k b_n < 0$ for $n \geq 1$.

Then

$$ D(z) := \sum_{j=1}^{N} a_j^0 p_j T (z^{l_j} - 1) - \sum_{k=1}^{K} \frac{1}{\sigma_k^2} B_k(z). \tag{37} $$

In this case, rate

$$ c_0 = -\sum_{j=1}^{N} a_j^0 p_j T - \sum_{k=1}^{K} \frac{1}{\sigma_k^2} \cdot {}_k b_0 , \tag{38} $$

which stands before $z^0$, is negative, whereas rates

$$c_n = \sum_{\{j:l_j=n\}} a_j^0 p_j T - \sum_{k=1}^{K} \frac{1}{\sigma_k^2} \cdot_k b_n \tag{39}$$

standing before $z^n$ are nonnegative.

Since $G_{\frac{X}{v}}(z) = \exp(D(z))$, so using lemma 1 again, it is possible to appoint expressions of this function. Let $d_n$ mean rates in the series expansion for $G_{\frac{X}{v}}$. Then

$$d_0 = \exp(c_0), \tag{40}$$

and

$$d_n = \sum_{k=1}^{n} \frac{k}{n} d_{n-k} c_k . \tag{41}$$

Since $c_0 < 0$ and $c_n \geq 0$ for every $n > 0$, so it is easy to show that $d_0 \geq 0$. Moreover it is possible also to determine $d_n$ as the sum of positive numbers. The result, which we receive, is our sought solution, i.e.:

$$d_n = P[X = nv]. \tag{42}$$

Algorithm using inverse Fourier transform

Since we don't need to consider individuals loss units, it is a great advantage of this climb. Moreover the numerical stability won't depend now on the structure of the characteristic function. Applying this method we will obtain neither densities nor distribution functions of the random variable *X*, but only the integral from the distribution function. So that, to use this method, it must be known expected values of the random variable *X*. Knowing this expected value is needed because the idea of this approach is to make the approximation of unknown distribution *X* by distribution of another random variable but with the same expected value. Since the loss *X* is a nonnegative random variable, therefore, also the approximated distribution should be determined on the set $\mathbf{R}^+$. However, the most often it is used the Gamma distribution approximation with the parameters $\alpha$ and $\beta$, which can be determined from the formulas:

$$\alpha = \frac{\mathrm{E}[X]^2}{Var[X]} \text{ and } \beta = \frac{\mathrm{E}[X]}{Var[X]}. \tag{43}$$

**Theorem 1.** [2]

*Let F and G be distribution functions of two random variables, appropriately X and Y, for which the third absolute moment exists. Let* $\varphi_X$ *and* $\varphi_Y$ *be their characteristic functions. Let establish, that* $\mathrm{E}[X] = \mathrm{E}[Y]$. *Let*

$$\underset{x \in \mathrm{R}}{\forall} \quad \hat{F}(x) := \int_{-\infty}^{x} F(y)\,dy \tag{44}$$

*and*

$$\underset{x \in \mathrm{R}}{\forall} \quad \hat{G}(x) := \int_{-\infty}^{x} G(y)\,dy. \tag{45}$$

*Then an equality is occurring*

$$\hat{F}(x) = \hat{G}(x) - \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-isx} \cdot \frac{\varphi_X(s) - \varphi_Y(s)}{s^2}\,ds. \tag{46}$$

Now we use this theorem. Let $F$ be the distribution function of random variable $X$, which describes credit loss from all portfolio, $G$ be Gamma distribution function, $\hat{F}$ and $\hat{G}$ be given by (44) and (45). Then:

$$\hat{F}(x) = \hat{G}(x) - \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-isx} \cdot \frac{\psi_X(s) - \left(\frac{\beta}{\beta - is}\right)^{\alpha}}{s^2}\,ds, \tag{47}$$

where

$$\hat{G}(x) = \int_{-\infty}^{x} G(t)\,dt = \int_{-\infty}^{x}\left(\int_{0}^{t} \frac{\beta^{\alpha} s^{\alpha-1} e^{-\beta \cdot s}}{\Gamma(\alpha)}\,ds\right)dt = \int_{0}^{x}\left(\int_{s}^{x} \frac{\beta^{\alpha} s^{\alpha-1} e^{-\beta \cdot s}}{\Gamma(\alpha)}\,dt\right)ds$$
$$= \int_{0}^{x} \frac{\beta^{\alpha} s^{\alpha-1} e^{-\beta \cdot s}}{\Gamma(\alpha)} \cdot (x - s)\,ds. \tag{48}$$

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[2] O. Reiss, *Fourier inversion algorithms for Generalized CreditRisk+ models and an extension to incorporate market risk*, WIAS-Preprint No. 817, 2003.

Substituting $\beta \cdot s = u$ , we get

$$\hat{G}(x) = \int_0^{\beta \cdot x} \frac{\beta^{\alpha} u^{\alpha-1} e^{-u}}{\beta^{\alpha-1} \cdot \Gamma(\alpha)} \cdot \left( x - \frac{u}{\beta} \right) \frac{du}{\beta} = x \cdot \int_0^{\beta \cdot x} \frac{u^{\alpha-1} e^{-u}}{\Gamma(\alpha)} du - \int_0^{\beta \cdot x} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+1)} \cdot \frac{u^{\alpha} e^{-u}}{\beta \cdot \Gamma(\alpha)} du . \tag{49}$$

So that, function $\hat{G}$ May be described as

$$\hat{G}(x) = x \cdot P(\alpha, \beta \cdot x) - \frac{\Gamma(\alpha+1)}{\beta \cdot \Gamma(\alpha)} \cdot P(\alpha+1, \beta \cdot x) = x \cdot P(\alpha, \beta \cdot x) - \frac{\alpha}{\beta} \cdot P(\alpha+1, \beta \cdot x), \tag{50}$$

where

$$P(\alpha, \beta \cdot x) = \int_0^{\beta \cdot x} \frac{u^{\alpha-1} e^{-u}}{\Gamma(\alpha)} du . \tag{51}$$

To receive a distribution function for *X*, we should count derivative from the function described by formula (47).

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**PORÓWNANIE METOD WYZNACZANIA ROZKŁADU STRATY Z PORTFELA KREDYTOWEGO W MODELU CREDITRISK+**

Model CreditRisk+ jest jedną z metod portfelowych służących do zarządzania ryzykiem kredytowym. W pracy omówione zostały założenia modelu, metody wyznaczenia oczekiwanych strat, jak również rozkładu straty z całego portfela kredytowego. Pierwsza numeryczna metoda stworzona przez Credit Suisse First Boston w 1997, która próbowała opisać ten model, bazowała na wzorze Panjer'a. Obecnie powstało kilka innych algorytmów umożliwiających wyznaczenie dystrybuanty straty z portfela kredytowego. Dwa z nich zostały omówione i porównane w tej pracy. Jeden algorytm bazuje na funkcji generującej prawdopodobieństwo, natomiast drugi wykorzystuje odwrotną transformatę Fouriera.