A METHOD OF COMPUTING MOMENTS
OF THE DURBIN-WATSON STATISTIC FOR LINEAR TRENDS

1. INTRODUCTION

To verify the hypothesis concerning the lack of autocorrelation in an econometric model
\[ y = X_1 u + \varepsilon \]  
(1)
the Durbin-Watson test is usually used because of simple calculations and quite a big power as well. The applications of the test, however, are limited because the so-called "nonconclusivity interval" does exist. It is connected with the fact that the distribution of the Durbin-Watson statistic depends on the matrix \( X \), thus tabulating of the critical values is practically impossible. Commonly known tables contain only lower and upper limits of the quantiles corresponding with the most frequently applied significance levels.

There is a possibility, however, to build such tables when \( X \) matrix is fixed. The table for the linear trend case when
\[
X = \begin{bmatrix}
1 & 1 \\
1 & 2 \\
1 & 3 \\
\vdots & \vdots \\
1 & n
\end{bmatrix}
\]  
(2)
is shown in the paper by Tomaszewicz [4].

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2. THE PROBLEM

An other possibility of investigating the Durbin-Watson statistic distribution which seems to be worth taking into account is to find its moments or cumulants. This paper presents general formulae for computing moments of the statistic for linear trend model. The formulae for the mean and standard deviation are shown as well.

We consider econometric model (1) (where $X$ has the form (2)) which fulfills classical assumptions i.e. $\varepsilon$ is of $n$-dimensional normal distribution:

$$\varepsilon \sim N(0, \sigma^2 I).$$

The Durbin-Watson statistic is given by the formula

$$d = \frac{\sum_{t=2}^{n} (e_t - e_{t-1})^2}{\sum_{t=1}^{n} e_t^2} = \frac{e^T A e}{e^T e}$$

where

$$A = \begin{bmatrix}
1 & -1 & 0 & \ldots & 0 & 0 \\
-1 & 2 & -1 & \ldots & 0 & 0 \\
0 & -1 & 2 & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & 2 & -1 \\
0 & 0 & 0 & \ldots & -1 & 1
\end{bmatrix}$$

and $e$ is the vector of residuals $e_t$ obtained by the ordinary least squares method:

$$e = y - Xa$$

where

$$a = (X^T X)^{-1} X^T y.$$
$e = Me$

where

\[ M = I - X(X'X)^{-1}X' \quad (3) \]

is very well known idempotent matrix. Further on we shall also use the symbols:

\[ u = e'Te, \quad v = e'e, \]

so

\[ d = \frac{u}{v}. \]

On the basis of von Neumann's result [3] Durbin and Watson [1] proved that the variables \( d = u/v \) and \( v \) are independent. Consequently, for given \( k \) we have

\[ Eu^k = Ed^k v^k = Ed^k Ev^k, \]

hence

\[ Ed^k = \frac{Eu^k}{Ev^k}. \quad (4) \]

To find \( d \) statistic moments it is enough to calculate the moments \( Eu^k \) and \( Ev^k \). Durbin and Watson [1] found formulae for moments \( d \) on the basis of the sums of the form

\[ \sum_{t=1}^{n-k_1} v_t^q = tr(MA)^q. \]

In particular

\[ Ed = \frac{1}{n-k_1} \sum_{t=1}^{n-k_1} v_t, \quad (5) \]

\[ Evd = \frac{2}{(n-k_1)(n-k_1 + 2)} \left( \sum v_t^2 - \frac{1}{n-k_1} \left( \sum v_t \right)^2 \right). \quad (6) \]
where
\[ \sum_{t} \nu_t = \text{tr}(MA) = \text{tr}A - \text{tr} X^TAX(X^TX)^{-1}, \]
\[ \sum_{t} \nu_t^2 = \text{tr}(MA)^2 = \text{tr}A^2 - 2\text{tr}(X^TAX(X^TX)^{-1}) \]
\[ + \text{tr}(X^TAX(X^TX)^{-1})^2. \]

3. FORMULAE FOR LINEAR TREND

We present formulae (4), (5) and (6) for the case of the linear trend model in which the \( X \) matrix is given by (2).

Let us denote
\[ M = I - N, \quad N = X(X^TX)^{-1}X^T. \]

Therefore
\[ (MA)^k = (A - NA)^k = \sum_{h=0}^{k} (-1)^h \binom{k}{h} A^{k-h} (NA)^h. \]

As from the very well known properties of trace
\[ \text{tr} A^{k-h} (NA)^h = \text{tr} A^{k-h} \underbrace{NA \ldots NA}_{h \text{ times}} = \]
\[ = \text{tr} A^{k-h} X(X^TX)^{-1} X^TAX(X^TX)^{-1} X^T \ldots X(X^TX)^{-1} X^T = \]
\[ = \text{tr} X A^{k-h+1} X(X^TX)^{-1} (X^TAX(X^TX)^{-1})^{h-1}. \]

Thus,
\[ s_k = \text{tr}(MA)^k = \text{tr} A^k + \]
\[ + \sum_{h=1}^{k} (-1)^h \binom{k}{h} \text{tr} A_{k-h} (X^TX)^{-1} (B_1(X^TX)^{-1})^{h-1} \quad (7) \]
where
\[ B_i = X^T A_i X \]
for \( i = 1, 2, \ldots, k \). Let
\[ X = [j x]. \]
Of course \( A_j = 0 \), hence each matrix \( B_k \) for \( k \geq 1 \) is of the form
\[ B_k = \begin{bmatrix} 0 & 0 \\ 0 & x^T A_k x \end{bmatrix}. \]
Therefore the first row of each of \( B_k (X^T X)^{-1} \) matrices consists of zeroes; thus, the trace of the product
\[ B_k (X^T X)^{-1} (B_i (X^T X)^{-1})^h \]
is equal to the product of diagonal elements of the second row of each element:
\[ \text{tr} B_k (X^T X)^{-1} (B_i (X^T X)^{-1})^h = b_k b_i^h, \]
where
\[ b_k = \frac{12}{n (n - 1) (n + 1)} x^T A_k x. \]
However, for \( k \geq 2 \)
\[ x^T A_k x = \frac{2}{k-1} \binom{2(k-2)}{k-2}, \]
thus, finally
\[ b_k = \frac{24}{n(n-1)(n+1)(k-1)} \binom{2(k-2)}{k-2} \]
for \( k \geq 2 \) and
\[ b_1 = \frac{12}{n (n + 1)} \]
because
\[ x^T A x = n - 1. \]

Taking into account (8), (9) and the formula
\[ \text{tr} A^k = n \frac{(2k - 1)!!}{k!} 2^k - 2^{2k-1}, \]

(7) can be rewritten in the form
\[ s_k = \text{tr}(MA)^k = n \frac{(2k - 1)!!}{k!} 2^k - 2^{2k-1} + \]
\[ + \sum_{h=1}^{k} (-1)^h \binom{k}{h} \frac{1}{(n-1)(k-h)} \]
\[ + \frac{1}{n-1} \sum_{h=1}^{k} (-1)^h \binom{k}{h} \frac{1}{k-h} \frac{2(k-h-1)}{k-h-1} \frac{24}{n(n+1)}^h. \]  

In particular
\[ s_1 = \text{tr} MA = \text{tr} A - b_1 \]
\[ + 2(n-1) - 2(n-1) \frac{12}{n(n-1)(n+1)} \]
\[ = 2(n-1) \left( 1 - \frac{12}{n(n-1)(n+1)} \right), \]  

(11)
\[ s_2 = \text{tr} A^2 - 2b_2 + b_1^2 \]
\[ = 2(3n-4) - \frac{48}{n(n+1)(n-1)} + \frac{576}{n^2(n+1)^2}. \]

Values \( s_k \) (10) put in the formulae for the moments \( m_A \) of the numerator \( u \) distribution can be expressed as functions of its cumulants \( x_k(u) \) (see: Durbin and Watson [1], p. 99-106 also Kendall and Stuart [2], p. 511-512).
\[ x_k(u) = 2^{k-1} (k-1)! s_k. \]

Particularly
\[ x_1(u) = s_1, \quad x_2(u) = 2s_2, \]
so
\[ m_1 = EU = x_1(u) = s_1, \quad (12) \]
\[ m_2 = EU^2 = x_2(u) - x_1(u)^2 = 2s_2 + s_1^2. \]

The moments of the denominator which is of \( \chi^2 \) distribution with \( n - 2 \) degrees of freedom are given by the formula
\[ Ev^k = \frac{(n-4+2k)!}{(n-4)!}. \]

Especially
\[ Ev = n - 2, \quad (13) \]
\[ Ev^2 = (n - 2)n. \]

It is easy to obtain from (4) and (11)-(13)
\[ Ed = \frac{s_1}{n - 2} = 2 \frac{n - 1}{n - 2} \left(1 - \frac{12}{n(n-1)(n+1)}\right) = \frac{20}{n(n+1)(n-2)} \]

and
\[ \nu^2d = \frac{EU^2}{Ev^2} - (Ed)^2 \]
\[ = \frac{2s_1 + s_2}{n(n-2)} - \frac{s_1^2}{(n-2)^2} \]
\[ = \frac{2s_2}{n(n-2)} - \frac{2s_1}{n(n-2)^2}. \]
Moments of the Durbin-Watson statistics:
lower $DL$, upper $DU$ and for linear trend $DT$

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<th>Std. deviation</th>
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As it can be seen, the Durbin-Watson statistic moments even in the easiest case of the linear trend model are expressed by quite complicated formulae, which are difficult to be applied in further analytic research. Nevertheless, using those formulae largely facilitates the numerical analysis in comparison with direct calculations \( tr(MA)^k \). That procedure was applied to find the first two moments of \( d \) statistics for some sample sizes. The results are shown in Table 1. Of course, the same procedure can be applied to compute higher-order moments.

REFERENCES


Andrzej S. Tomaszewicz

O PÆWÆJÆ MÆTOĐÆ OBLICZANIA
MOMEMÆ STÆSTYKI DÆURBINA-WATÆSONA
ZWIÀÆJANEÆ Z LINDÆOWÆM TRENDÆM

Celem artykułu jest opis pewnej numerycznie atrakcyjnej metody obliczania momentów statystyki testu Durbin-Watsona dla hipotezy o braku autokorelacji w liniowym modelu trendu.

Durbin i Watson [1] podali sposób obliczania momentów tej statystyki w zależności od sum wartości własnych potęg macierzy MA.
Autorowi udało się znaleźć wyrażenie, za pomocą którego można te sumy wyznaczyć.

Metoda została zastosowana do obliczenia dwóch pierwszych momentów. Wyniki są zawarte w tab. 1.