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Chapter 21
Topological and algebraic aspects of subsums of series

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The investigation of topological properties of sets of subsums for absolutely convergent series has been initiated almost one hundred years ago by Soichi Kakeya [11], [12]. A major step in the research took place in 1988 when J.A. Guthrie and J.E. Nymann published the full topological classification of the sets of subsums [8] – wider than Kakeya thought. However, their theorem is ineffective in the sense that it lists all four possible (up to homeomorphisms) types of sets of subsums, but provides no tool for recognition of the type for a given series. Finding a complete analytic characterization of the Guthrie-Nymann classification remains a challenging problem and we present the current state of research in this direction. Starting with a new exposition of the Guthire-Nymann Classification Theorem (based upon [21]), we survey all known examples of series leading to M-Cantorvals together with very recently discovered sufficient conditions for such series. The topological classification of the sets of subsums induces a natural division of the classic Banach space l_1 into four disjoint sets. Interesting algebraic and topological properties of the division are also discussed in the survey.

21.1 Sets of subsums of series

Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of real numbers tending to zero. For a series $\Sigma_{n\in\mathbb{N}}a_n$ and a given a given set $B\subset\mathbb{N}$, we will say that the series: $\sum_{n\in B} a_n$ is a **subseries** of $\sum_{n\in\mathbb{N}} a_n$. If B is finite, we will say that $\sum_{n\in B} a_n$ is a **finite subseries** of $\sum_{n\in\mathbb{N}} a_n$. We agree to write $\sum_{n\in\emptyset} a_n = 0$. We are going to investigate **the set of subsums of a series**, that is, the set

$$E = E(a_n) := \left\{ x\in\mathbb{R} :\ \exists\, B\subset\mathbb{N} \quad \sum_{n\in B} a_n = x \right\}.$$

We will also write

$$E = \left\{ \sum \varepsilon_n a_n :\ \varepsilon_n \in \{0,1\} \right\},$$

assuming tacitly, that we consider such choises of $(\varepsilon_n)_{n\in\mathbb{N}}$ only that lead to convergent subseries.

The restricted definition allows for a very nice, transparent and natural classification of series. We start with a classification of series from the point of view of their behaviour under rearrangements. We will say that a series $\sum a_n$ is **strongly divergent** if $\sum a_{\pi(n)}$ diverges for every permutation π of its terms. We will say that a series $\sum a_n$ is **absolutely convergent** if $\sum a_{\pi(n)}$ converges for every permutation π of its terms. We know from the elementary theory of series that $\sum a_n$ is absolutely convergent if and only if $\sum |a_n|$ converges ($(a_n)\in l_1$). Other series which are neither strongly divergent nor absolutely convergent will be called **potentially non-absolutely convergent**. These are extactly series $\sum a_n$ for which there are permutations π_1 and π_2 of $\mathbb{N}$ such that $\sum a_{\pi_1(n)}$ converges and $\sum a_{\pi_2(n)}$ diverges. Thus the potentially non-absolutely convergent series are exactly the series that are non-absolutely convergent or can be rearranged into a non-absolutely convergent series.

All three classes of series defined above have very transparent characterizations in terms of subseries of all positive terms and of all negative terms. With the classic definitons

$$a_n^+ := \max\{a_n, 0\} \qquad \text{and} \qquad a_n^- := \max\{-a_n, 0\},$$

we get the following well known characterizations.

Theorem 21.1. *A series $\sum a_n$ is absolutely convergent if and only if both series $\sum a_n^+$ and $\sum a_n^-$ converge.*

A series $\sum a_n$ is potentially non-absolutely convergent if and only if both series $\sum a_n^+$ and $\sum a_n^-$ diverge.

A series $\sum a_n$ is strongly divergent if and only if exactly one of the series $\sum a_n^+$ and $\sum a_n^-$ converges.

Another characterization of our classification of series can be given in terms of their sets of subsums. We need an auxilliary fact (which would be false if not the initial agreement that general terms of a series must tend to 0) (cf. [3]).

Lemma 21.2. *If $\sum a_n$ is a divergent series of positive terms then every positive number is the sum of an infinite subseries of $\sum a_n$ and hence $E(a_n) = [0, +\infty)$.*

We are ready for a theorem that tells us how to recognize the type of a series by looking at its sets of subsums $E = E(a_n)$.

Theorem 21.3. *A series $\sum a_n$ is:*
(i) strongly divergent if and only if the set E is a half-line.
(ii) potentially non-absolutely convergent if and only if $E = \mathbb{R}$.
(iii) absolutely convergent if and only if E is bounded.

Proof. Since our classification of series forms a division of the set all series, it suffices to prove implications from the left to the right in all three cases.

First, consider the case $A := \sum a_n^+ < +\infty$ and $\sum a_n^- = -\infty$. Applying Lemma 21.2 to the series $\sum(-a_n^-)$, we conclude that its sets of subsums is the half-line $(-\infty, 0]$. It follows that $E(\sum a_n) = (-\infty, A]$. The proof in the case $\sum a_n^+ = +\infty$ and $B := -\sum a_n^- > -\infty$ is analogous and leads to the conclusion that $E := [B, \infty)$ which completes the proof of left-to-right implication in (i).

Next, if the series $\sum a_n$: is potentially non-absolutely convergent then using the Lemma 21.2 to both series $\sum a_n^+$ and $\sum(-a_n^-)$ and taking into account the the sum of an empty subseries is 0, we obtain $E = \mathbb{R}$.

Finally, if the values $A = \sum a_n^+$ and $B = -\sum a_n^-$ both are finite, we get $E \subset [B, A]$. □

If an absolutely convergent series $\sum a_n$ has only finitely many non-zero terms, then $E = E(a_n)$ is a finite subset of $\mathbb{R}$ and therefore presents no topological mysteries whatsoever. On the other hand, the removal of all zero terms from any series does not change their sets of subsums. Therefore we may and from now on we will assume that all terms of the investigated sequences (a_n) are non-zero. Even more, we are now going to show that in order to describe all topological properties of sets of subsums of absolutely convergent series it suffices to consider only series of positive terms. Indeed, let α be the sum of all positive terms of a series $\sum a_n$ and let β be the sum of all negative terms, that is,

$$\alpha := \sum_{a_n>0} a_n \qquad \text{and} \qquad \beta := \sum_{a_n<0} a_n.$$

Next, let E' be the set of subsums of the series of absolute values of terms of $\sum a_n$. Finally, given a subset $A \subset \mathbb{N}$, denote

$$A_+ := \{n \in A : \ a_n > 0\} \qquad \text{and} \qquad A_- := \{n \in A : \ a_n < 0\}.$$

Now, if the series $\sum a_n$ is absolutely convergent, we may use associativity and commutativity of infinite addition freely and hence

$$\begin{aligned}\sum_{n\in A} a_n &= \sum_{n\in A_+} a_n + \sum_{n\in A_-} a_n = \sum_{n\in A_+} |a_n| + \sum_{\substack{a_n<0\\ n\notin A_-}} |a_n| + \sum_{a_n<0} a_n \\ &= \sum_{\substack{n\in A_+\\ \text{or}\\ (a_n<0 \text{ and } n\notin A_-)}} |a_n| + \beta \in \ E' + \beta.\end{aligned}$$

Thus, : $E \subset E' + \beta$. On the other hand,

$$\begin{aligned}\sum_{n\in A} |a_n| &= \sum_{n\in A_+} a_n - \sum_{n\in A_-} a_n = \sum_{n\in A_+} a_n - \sum_{a_n<0} a_n + \sum_{\substack{a_n<0\\ n\notin A_-}} a_n \\ &= \sum_{\substack{n\in A_+\\ \text{or}\\ (a_n<0 \text{ and } n\notin A_-)}} a_n - \beta \in \ E - \beta.\end{aligned}$$

Thus $E' \subset E - \beta$ and therefore $E \ = \ E' \ + \ \beta$. In particular, the sets E and E' are homeomorphic.

Actually, the investigation of the topological type of sets of subsums of an absolutely convergent series can be reduced even further. We can assume that the investigated series not only has all terms positive, but also that its terms are arranged in non-increasing manner and that sum of the series is 1. Indeed, if $E = E(a_n)$, then defining

$$\tilde{a}_i \ := \ \frac{a_i}{\sum_{n=1}^{\infty} a_n},$$

we obtain a series $\sum \tilde{a}_n$ of sum 1. Further, denoting $\tilde{E} \ := \ E(\tilde{a}_n)$, we get

$$E(a_n) \ = \ E\left(\left(\sum_{n=1}^{\infty} a_n\right)\tilde{a}_i\right) \ = \ \left(\sum_{n=1}^{\infty} a_n\right)\cdot \tilde{E},$$

that is E is the image of $\tilde{E}$ under a dilation. In particular, the sets E and $\tilde{E}$ are homeomorphic.

Summarizing, if we investigate topological properties of the set of subsums of an absolutely convergent series, we may, with little loss of generality, assume that the considered series is of positive non-increasing terms and of sum 1. In that case the set E of subsums becomes the set of values of a purely atomic probabilistic measure. In fact, there are a number of papers using such a language: [15], [4], [7]. The mentioned loss of generality is caused by omitting the case of almost all terms equal to zero, but, as we easily see, this is equivalent to the case when E is a finite set.

Therefore, from now on until the end of this section whenever we will discuss a series $\sum a_n$ without any explicit assumptions, we will assume that the series has positive and non-increasing terms and that its sum is 1.

Theorem 21.4. *The set $E(a_n)$ is closed for every absolutely convergent series $\sum a_n$.*

It is one of classical results of Kakeya [11], [12] (see also [20], Problems 130-132) and it was rediscovered later in [9] and [14].

We turn now towards a number of simple but important notions that will give us a better look into the structure of sets of subsums. Given an index $k \in \mathbb{N}_0$, we denote

$$E_k := \left\{ \sum_{n=k+1}^{\infty} \varepsilon_n a_n : \forall n \quad \varepsilon_n \in \{0, 1\} \right\}.$$

Thus, E_k is the set of subsums of the k-th remainder of the series $\sum a_n$. The value of the k-th remainder will be denoted by r_k. In particular, $E_0 = E(a_n)$. The set of all k-initial subsums of $\sum a_n$ will be denoted by

$$F_k := \left\{ \sum_{n=1}^{k} \varepsilon_n a_n : \forall n \in \{1, \ldots, k\} \quad \varepsilon_n \in \{0, 1\} \right\}.$$

We define $F_0 := \{0\}$ additionally.

The following fact tells us that the set E is a union of finitely many translates of the set of subsums of the k-th remainder.

Fact 21.5. For any $k \in \mathbb{N}$, the following equalities hold

$$E_{k-1} = E_k \cup (a_k + E_k)$$

and

$$E \;=\; \bigcup_{f\in F_k} (f + E_k).$$

Sometimes we will need a list of all elements of F_k in the increasing order:

$$0 \;=\; f_1^{(k)} \;<\; f_2^{(k)} \;<\; \cdots \;<\; f_{t(k)}^{(k)} \;=\; \sum_{n=1}^{k} a_n,$$

where $t(k) := |F_k|$ (the cardinality of F_k). Clearly, $k+1 \le t(k) \le 2^k$ always (we keep assuming that $\sum a_n$ is a convergent series of positive non-increasing terms and of sum 1; in particular, $f_2^{(k)} = a_k$ for any $k \in \mathbb{N}$).

Fact 21.6. The set $F := \bigcup_k F_k$ of all sums of finite subseries is dense in E.

Proof. Clearly, $F \subset E$, and hence $\overline{F} \subset \overline{E}$. Thus, $\overline{F} \subset E$ by the Thm. 21.4. On the other hand, if $x \in E$, then

$$x \;=\; \sum_{n\in A} a_n \;=\; \lim_{k\to\infty} \sum_{\substack{n\in A\\ n\le k}} a_n \;\in\; \overline{F},$$

that is, $E \subset \overline{F}$. □

Another of classic Kakeya's results is the following (see [11], [12], [9], [14]).

Theorem 21.7. *The set E has no isolated points (hence E is always a perfect set).*

It is easy to see that the set E always is symmetric with respect to the point $\frac{1}{2}$. The next fact provides a rather natural description of E as an intersection of a descending family of finite unions of closed intervals. Given a series $\sum a_n$ and a non-negative integer k, the set

$$I_k := \bigcup_{f\in F_k} \big(f + [0, r_k]\big)$$

will be called **the k-th iterate of the set E**.

Fact 21.8.

$$E = \bigcap_{k=1}^{\infty} \bigcup_{f\in F_k} \big(f + [0, r_k]\big) = \bigcap_{k=1}^{\infty} I_k.$$

Proof. Given a non-negative integer k, we get by the Fact 21.5 that

$$E = \bigcup_{f \in F_k} (f + E_k) \subset \bigcup_{f \in F_k} (f + [0, r_k]) = I_k.$$

Thus $E \subset \bigcap_k I_k$. On the other hand, if $x \in \bigcup_{f \in F_k} (f + [0, r_k])$, then $d(x, F_k)$, that is the distance of x to F_k, does not exceed r_k. Hence, if $x \in \bigcap_k I_k$, then $d(x, F) = 0$. It means $x \in \overline{F} = E$ by the Fact 21.6 which completes the proof of the reverse inclusion $\bigcap_k I_k \subset E$. □

A series $\sum a_n$ is said to be **quickly convergent** if $a_n > r_n$ for all n. The terminology has been introduced in [17].

Fact 21.9 (The First Gap Lemma). If $a_k > r_k$ for some index k, then the open interval (r_k, a_k) is a gap of E (ie a component of the complement of E).

Proof. Clearly, both endpoints of (r_k, a_k) belong to E. Suppose that (r_k, a_k) is not a gap of E. Then $r_k < p < a_k$ for some $p \in E$. Since the terms of $\sum a_n$ are non-increasing, we have $p < a_k \leq a_{k-1} \leq \ldots \leq a_1$ as well. Thus, the k initial terms are excluded from any representation of p as a subsum of $\sum a_n$ and therefore p is a subsum of $\sum_{n=k+1}^{\infty} a_n$. In particular, $p \leq r_k$, a contradiction. □

Yet another of classic Kakeya's results is

Theorem 21.10. *If $\sum a_n$ is quickly convergent, then its set of subsums E is a Cantor set (that is, homeomorphic to the classic Cantor ternary set). Moreover, the Lebesgue measure of E is $\mu E = \lim_n 2^n r_n$.*

It is known that if $\sum a_n$ is quickly convergent, then
(i) x is the right endpoint of an E-gap if and only if $x \in F$.
(ii) x is the left endpoint of an E-gap if and only if $\exists\, k \in \mathbb{N} \quad x \in F_{k-1} + r_k$.
It follows from the Thm. 21.10 that a finite limit $\lim_n 2^n r_n$ exists for every quickly convergent series $\sum a_n$.

A series $\sum a_n$ is said to be **slowly convergent** if $a_n \leq r_n$ for all n.

Let us recall that if the set F_k of all subsums using at most k initial terms of the series $\sum a_n$ is listed in the increasing order of elements then we use the symbol $f_j^{(k)}$ for the j-th term of the increasing finite sequence (see page 350). The following Fact provides a vague description of endpoints of E-gaps without the assumption that the underlying series $\sum a_n$ is quickly convergent.

Fact 21.11 (The Second Gap Lemma). Let (a, b) be an E-gap. Define $k := \max\{n : \ a_n \geq b - a\}$. Then $b \in F_k$. Moreover, if $b = f_j^{(k)}$, then $a = f_{j-1}^{(k)} + r_k$.

Proof. Observe first that $b \in F_k$. Otherwise, every representation of the form $b = \sum_{n\in A} a_n$ must involve at least one term a_l with $l > k$. Then $b - a_l \in E$ and, by the definition of k, $b - a_l \in (a, b)$ contradicting the assumption that (a, b) is an E-gap. Thus, $b \in F_k$, indeed.

Recall that $E = \bigcup_{f\in E_k}(f + F_k)$, where $E_k = E(\sum_{n=k+1}^{\infty} a_n)$ and $F_k = \{f_j^{(k)} : j = 1, 2, \dots, t(k)\}$. Let j be such that $b = f_j^{(k)}$. Then $f_{j-1}^{(k)} \leq a$, since $(a, f_j^{(k)})$ is an E-gap.

Suppose now that $f_{j-1}^{(k)} + r_k > b$. Then $(f_{j-1}^{(k)} + r_i)_{i=k}^{\infty}$ is a sequence decreasing to $f_j^{(k)} \leq a$ such that the difference between any two consequtive terms is less than $b - a$. Hence the interval (a, b) contains at least one term of the sequence and thus $E \cap (a, b) \neq \emptyset$, a contradiction. Therefore, we have

$$f_{j-1}^{(k)} + r_k \leq a. \tag{21.1}$$

On the other hand, $a < f_i^{(k)}$ for $i \geq j$. Hence $a \in \bigcup_{i<j}(f_i^{(k)} + E_k)$ which implies that $a \leq \sup(f_{j-1}^{(k)} + E_k) = f_{j-1}^{(k)} + r_k$. Thus, by (21.1), $a = f_{j-1}^{(k)} + r_k$. □

Example 21.12. Consider the series

$$a_1 = 1,\ a_2 = \frac{15}{16},\ a_3 = a_4 = \frac{1}{2},\ a_5 = \frac{7}{16},\ a_n = \frac{1}{2^n} \text{ for } n \geq 6.$$

We ought to multiply all terms of the series by a suitable factor in order to guarantee its sum is 1, but it is inessential for the example. The interval $(a, b) := (\frac{31}{32}, 1)$ is a gap of E. b has exactly two representations as a subsum of the series: $b = a_1$ and $b = a_3 + a_4$. Since $F_1 = \{0, 1\}$, we have $b \in F_1$ with $\varepsilon_1 = 1$. Moreover,

$$b = f_2^{(1)}, \qquad a = \frac{31}{32} < f_1^{(1)} + r_1 = 0 + 2\frac{13}{32}.$$

Also $b \in F_4$ with $\varepsilon_4 = 1$ and $F_4 = \{0, \frac{1}{2}, \frac{15}{16}, 1, 1\frac{7}{16}, 1\frac{1}{2}, 1\frac{15}{16}, 2, 2\frac{7}{16}, 2\frac{15}{16}\}$. Thus,

$$b = f_4^{(4)}, \qquad a = \frac{31}{32} < f_3^{(4)} + r_4 = \frac{15}{16} + 2\frac{15}{32}.$$

Hence a is not of the form $f_{j-1}^{(k)} + r_k$ for any representation $b = \sum_{i=1}^{k} \varepsilon_i a_i$ with $\varepsilon_k = 1$, where j is such that $b = f_j^{(k)}$. That is, one of statements in the initial part of the proof of Lemma 2, [18] is false. Fortunately, the mistake had no influence on corectness of results from the cited paper.

Here is the last of classic Kakeya theorems on partial sums that we want to recall. It was rediscovered not only by already metioned H. Hornich and P. Kesava Menon, but also by A. D. Weinstein and B. E. Shapiro in [23].

Theorem 21.13. $E = [0, 1]$ *if and only if the series* $\sum a_n$ *is slowly convergent.*

The above theorem has turned out to be the perfect tool for showing that every continuous measure has the Darboux property [22] which confirms the strong relationship of Kakeya's theorems to the basic measure theory.

Corollary 21.14. *E is a union of finitely many closed intervals if and only if $a_n \le r_n$ for all sufficiently large indices n.*

Proof. ($\Rightarrow$) If E is a union of a finite family of closed intervals, then $[0, 1] \setminus E$ is a union of a finite family of pairwise disjoint open intervals. Therefore, the lengths of gaps of E are bounded away from 0 and thus, by the First Gap Lemma, $a_n \le r_n$ for all sufficiently large n.

($\Leftarrow$) If $a_n \le r_n$ for $n > N$, then $E_N = [0, r_N]$ by the Thm. 21.13. Now, the equality $E = \bigcup_{f \in F_N} (f + E_N)$ completes the proof. □

We are now turning our attention towards a discussion of a bold hypothesis formulated by S. Kakeya. Namely, he thought that every set of subsums of a convergent series of positive terms is either a union of a finite family of bounded closed intervals or a Cantor set, and he wrote openly: "but I have no proof of it".

21.2 M-Cantorvals

The hypothesis remained open until the work of Weinstein and Shapiro who gave an example of a series with the set of subsums being neither of the two known to Kakeya types [23]. The series provided by Weinstein and Shapiro has an M-Cantorval as the set of its subsums. The full classification of sets of subsums up to homeomorphisms was eventually found by J. A. Guthrie and J. E. Nymann in [8] and their proof was essentially repaired by Nymann and Sáenz in [18]. We are going to present here the classification theorem with a new and short proof based on the Mendes-Oliveira characterization of M-Cantorvals. We are now turning towards the definition of an M-Cantorval.

Let us start from some basic notions. Connectivity components of a closed set $D \subset \mathbb{R}$ are either closed intervals or singletons. Intervals that are connectivity components of a closed set D will be called ***D*-intervals**, while one-point connectivity components of D will be called **loose points of** D. Open intervals that are connectivity components of D' will be called D**-gaps**. If D is bounded, then the two unbounded D-gaps will be called **exterior** D**-gaps**. Bounded D-gaps will be called **interior** D**-gaps**.

Let C be the classic Cantor set. The **order of an interior** C**-gap** is defined to be the number of step of the standard construction of the Cantor set in which the gap was removed from $[0,1]$. For instance, $(\frac{1}{3},\frac{2}{3})$ is a C-gap of order 1, $(\frac{7}{9},\frac{8}{9})$ is one of two C-gaps of order 2, $(\frac{19}{81},\frac{20}{81})$ is one of eight C-gaps of order 4. We do not assign any order to the exterior C-gaps.

We will say that a sequence (I_n) of intervals of $\mathbb{R}$ **converges** to a point $x \in \mathbb{R}$ if

$$\forall \varepsilon > 0 \quad \exists N \in \mathbb{N} \quad \forall\, n \geq N \qquad I_n \subset (x-\varepsilon, x+\varepsilon),$$

that is, if the closures of the intervals tend to the singleton : $\{x\}$: in the Vietoris topology. Observe three simple properties of the classic Cantor set C.

Fact 21.15. The classic Cantor set C enjoys the following properties:

(c1) Given any two C-gaps, there are an even order C-gap and an odd order C-gap between them.

(c2) Every point of C is the limit of a sequence of C-gaps of even order and of a sequence of C-gaps of odd order.

(c3) The set of endpoints of all odd order C-gaps is dense in C.

The **Guthrie-Nymann set** is defined to be

$$GN \;:=\; C \cup \bigcup_{n=1}^{\infty} G_{2n-1} \;=\; [0,1] \setminus \bigcup_{n=1}^{\infty} G_{2n},$$

where G_k denotes the union of all C-gaps of order k. Clearly, GN is a nonempty bounded perfect set, since no two GN-gaps have common endpoints. The GN-gaps are exactly C-gaps of even order. The GN-intervals are exactly closures of C-gaps of odd order. GN has infinitely many component intervals and therefore is not homeomorphic to the classic Cantor set.

How can we characterize all perfect subsets of $\mathbb{R}$ that are homeomorphic to GN? We start with a list of topological properties of the set GN.

Fact 21.16. The Guthrie-Nymann set enjoys the following properties:

(GN1) GN-gaps and GN-intervals have no common endpoints.

(GN2) Endpoints of all GN-gaps are limits of sequences of GN-intervals and limits of sequences of GN-gaps.

(GN3) Given any two GN-intervals (or any two GN-gaps, or a GN-gap and a GN-interval), there are a GN-interval and a GN-gap between them.

(GN4) The union of all GN-intervals is dense in GN.

Theorem 21.17. *A nonempty bounded perfect set $P \subset \mathbb{R}$ is homeomorphic to the Guthrie-Nymann set if and only if*

(i) P-gaps and P-intervals have no common endpoints

and

(ii) the union of all P-intervals is dense in P.

Proof. The necessity of both conditions (i) and (ii) follows easily from properties (GN1) and (GN4).

Assume now that $P \subset \mathbb{R}$ is a nonempty bounded perfect set with properties (i) and (ii) and denote $a := \inf P$, $b := \sup P$. We are going to construct a homeomorphism $h: [0, 1] \to [a, b]$ (an increasing continuous surjection) such that $h(GN) = P$. Let $(I_i^{GN})_{i\in\mathbb{N}}$ be a joint enumeration of all GN-intervals and all closures of interior GN-gaps. Analogously, let $(I_i^P)_{i\in\mathbb{N}}$ denotes a sequence of all P-intervals and closures of all interior P-gaps. It follows from the property (i) that the last sequence is infinite, indeed.

We are ready for inductional construction of a function $f: \bigcup_i I_i^{GN} \to \bigcup_i I_i^P$. Take the interval I_1^{GN}. If it is a GN-interval, then we map it in the linear increasing manner onto the first P-interval in the sequence $(I_i^P)_{i\in\mathbb{N}}$. If I_1^{GN} is the closure of a bounded GN-gap, then we define $f\big|_{I_1^{GN}}$ to be the increasing linear map of the interval onto the first closure of a P-gap in the sequence $(I_i^P)_{i\in\mathbb{N}}$.

Suppose now that n is a positive integer such that there is an increasing continuous injection f of $\bigcup_{i=1}^n I_i^{GN}$ into $\bigcup_i I_i^P$ such that f takes GN-intervals onto P-intervals and takes closures of GN-gaps onto closures of P-gaps. Consider the interval I_{n+1}^{GN}. Exactly one of the following cases holds:

(a) I_{n+1}^{GN} lies between the intervals I_i^{GN} and I_j^{GN} for some i, $j \le n$.

(b) I_{n+1}^{GN} lies to the right of all I_i^{GN} for $i = 1, \ldots, n$.

(c) I_{n+1}^{GN} lies to the left of all I_i^{GN} for $i = 1, \ldots, n$.

In the case (a), if I_i^{GN} is a GN-interval (the closure of a GN-gap), then we map it in the linear and increasing manner onto the P-interval (the closure of a P-gap) with the smallest index in the sequence $(I_i^P)_{i\in\mathbb{N}}$ among indices of all P-intervals (of all closures of a P-gaps) lying between $f(I_i^{GN})$ and $f(I_j^{GN})$. In the case (b), if I_i^{GN} is a GN-interval (the closure of a GN-gap), then we map it in the linear and increasing manner onto the P-interval (the closure of a P-gap) with the smallest index in the sequence $(I_i^P)_{i\in\mathbb{N}}$ among indices of all P-intervals (of all closures of a P-gaps) lying to the right of all $f(I_i^{GN})$ for $i = 1, \ldots, n$. The case (c) is fully analogous to the case (b).

This construction yields an increasing continuous surjection f of $\bigcup_i I_i^{GN}$ onto $\bigcup_i I_i^P$ such that the image under f of the union of all GN-intervals is the union of all P-intervals.

Let us recall that a bounded and increasing continuous function $g : A \to \mathbb{R}$ defined on a set A dense in a closed interval $[\alpha, \beta]$ can be extended to a continuous function $\overline{g} : [\alpha, \beta] \to \mathbb{R}$ if and only if

$$\lim_{x \to x_0^+} g(x) = \lim_{x \to x_0^-} g(x)$$

for every $x_0 \in (\alpha, \beta) \setminus A$. Then $\overline{g}$ is an increasing function from $[\alpha, \beta]$ onto $[\lim_{t\to\alpha^+} g(t), \lim_{t\to\beta^-} g(t)]$.

The constructed by us function f is defined on $[0, 1]$ except for the loose points of GN which are not endpoints of interior GN-gaps. Let x_0 be such an exceptional point. The function f is increasing and bounded and hence there exist finite limits $\lim_{x\to x_0^-} f(x) \le \lim_{x\to x_0^+} f(x)$. Suppose that the two limits are distinct. Then the open interval $\left(\lim_{x\to x_0^-} f(x), \lim_{x\to x_0^+} f(x)\right)$ has no common points with $f\left(\bigcup_i I_i^{GN}\right) = \bigcup_i I_i^P$ which contradicts the fact that the last set is dense in $[a, b]$. Hence, it must be $\lim_{x\to x_0^-} f(x) = \lim_{x\to x_0^+} f(x)$ which proves that f can be uniquely extended to a homeomorphism $h : [0, 1] \to [a, b]$.

It remains to show that $h(GN) = P$. Since h is a homeomorphism, we get $h(\overline{A}) = \overline{h(A)}$ for any $A \subset [0, 1]$. In particular, choosing A to be the union of all GN-intervals, we get $h(GN) = P$ by the property (ii) and by our construction of h. □

A set homeomorphic to the GN set will be called an **M-Cantorval**. Another characterization of M-Cantorvals was given by Mendes and Oliveira in [16].

Theorem 21.18. *A nonempty bounded perfect set $P \subset \mathbb{R}$ is an M-Cantorval if and only if all endpoints of P-gaps are limits of sequences of P-intervals and limits of sequences of P-gaps.*

Proof. A short outline of a direct constructional proof of the Mendes-Oliveira characterization of M-Cantorvals can be found in the Appendix of [16]. We are going to present here another proof based on Thm. 21.17.

Suppose that P is an M-Cantorval. Then it has properties (i) and (ii) of Thm. 21.17. Observe that if a, b (with $a < b$) are points of P such that

$(*)$ a, b are not endpoints of the same P-gap

and

$(**)$ a and b do not belong to the same P-interval,

then the open interval (a, b) contains a P-interval. Indeed, the interval (a, b) must contain a point of the complement of P by $(**)$. Hence, since $a, b \in P$, the interval must contain a P-gap. Now, at least one of the endpoints of the P-gap must lie in (a,b) by $(*)$. This endpoint cannot be an endpoint of a P-interval because of (i). Hence, by (ii), there is at least one P-interval contained in (a,b).

Let $x \in P$ be an endpoint of a P-gap. The point x does not belong to any P-interval by (i). On the other hand, since P is perfect, there is a sequence (x_n) of points of P monotonically convergent to x. Passing, if necessary, to a subsequence, we may assume that any two consecutive terms of the sequence are neither endpoints of the same P-gap nor belong to the same P-interval. Thus, by our earlier observation, there is a P-interval P_n between x_n and x_{n+1}. The sequence (P_n) converges to x and $P_i \cap P_j = \emptyset$ for $i \neq j$. If G_n denotes any P-gap lying between P_n and P_{n+1}, then the sequence (G_n) converges to x as well. Hence all endpoints of P-gaps are limits of sequences of P-intervals and limits of sequences of P-gaps.

Now, let $P \subset \mathbb{R}$ be a nonempty bounded perfect set such that all endpoints of P-gaps are limits of sequences of P-intervals and limits of sequences of P-gaps. This property implies instantly that a P-gap and a P-interval cannot have a common endpoint, that is, P satisfies the property (i) of Thm. 21.17.

Let x be a point of P not belonging to any of P-intervals. Take any sequence (x_n) of points of P monotonically convergent to x. Passing, if necessary, to a subsequence, we may assume that no two consequtive terms of the sequence belong to the same P-interval or are endpoints of the same P-gap. Therefore, given any positive integer n, there is a P-gap between x_n and x_{n+1} such that at least one of the endpoints of the gap belongs to the open interval with endpoints x_n and x_{n+1}. According to our assumption about P, the endpoint of the gap is a limit of a sequence of P-intervals. Hence the open interval with endpoints x_n and x_{n+1} contains infinitely many P-intervals. Choosing one of them and denoting it by P_n, we obtain a sequence (P_n) of P-intervals convergent to x. Hence x belongs to the closure of the union of all P-intervals. Since $x \in P$ was arbitrary, we conclude that the set P has the property (ii) of Thm. 21.17 as well.

Then P is an M-Cantorval by the Thm. 21.17. □

21.3 Sets of subsums of series and Cantorvals

The first essential appearance of an M-Cantorval popped up in the paper [23] and it was given as a counterexample to a hypothesis on sets of subsums of an absolutely convergent series. M-Cantorvals turned out to be one of four possible topological types of sets of subsums of an absolutely convergent series [8]. However Guthrie and Nymann did not use the name; they wrote about sets homeomorphic to the set T of subsums $\sum \beta_n$ where $\beta_{2n-1} = 3/4^n$ and

$\beta_{2n} = 2/4^n$ ($n = 1, 2, \ldots$). The Guthrie-Nymann set was given as a transparent example of a set homeomorphic to the set T in [8]. Finally, when Mendes and Oliveira characterized topological types of algebraic sums of homogeneous Cantor sets in [16], they defined various types of Cantorvals, including the M-Cantorvals, and used the name explicitly.

We need first a theorem that tells us that the set of subsums is locally identical near endpoints of its gaps and it will be the crucial tool in proving the topological classification of sets of subsums of absolutely convergent series. It was proved in [18] and a number of versions of it were developed in more general settings (Lemma 3.3, [19] and Proposition 2.1, [1]).

Theorem 21.19 (Nymann-Sáenz Theorem). *If (a, b) is an interior E-gap, then the following equalities hold*

$$b + \big([0, \varepsilon] \cap E\big) = [b, b + \varepsilon] \cap E$$

and

$$\big([1 - \varepsilon, 1] \cap E\big) - (1 - a) = [a - \varepsilon, a] \cap E$$

for all sufficiently small $\varepsilon > 0$.

Proof. We start with an **Observation 1**:

$$E \cap [0, \varepsilon] = E_k \cap [0, \varepsilon] \qquad \text{for } \varepsilon < a_k.$$

The inclusion $\supset$ above is obvious. On the other hand, if $x \in E$ and $x < a_k$, then x is the sum of some terms less than a_k, that is, some terms with indices greater than k. Hence, $x \in E_k$.

Observation 2: Let b be the right endpoint of an interior E-gap. Let k and $f_j^{(k)} = b$ be as in the Second Gap Lemma (Fact 21.11).

If $j = t(k)$ (see page 350), then $b = \sum_{n=1}^{k} a_n$. Hence if $x \in E$ and $x > b$, then $x \in b + E_k$ by the Second Gap Lemma. The inclusion $b + E_k \subset E$ is obvious. Hence $(b, +\infty) \cap E = b + E_k$ and thus

$$[b, b+\varepsilon] \cap E \;=\; [b, b+\varepsilon] \cap (b + E_k) \qquad \text{for every } \varepsilon > 0.$$

If $j < t(k)$, then taking $x \in [b, b+\varepsilon] \cap E$, where $\varepsilon < f_{j+1}^{(k)} - f_j^{(k)}$, and its representation $x = \sum_{n \in A} a_n$, we look at the trivial equality

$$x \;=\; \sum_{\substack{n \in A \\ n \le k}} a_n + \sum_{\substack{n \in A \\ n > k}} a_n.$$

Clearly, $\tilde{x} := \sum_{\substack{n\in A\\ n\le k}} a_n \in F_k$. If $\tilde{x} < b$, then $\tilde{x} \le f^{(k)}_{j-1}$, and $x \le f^{(k)}_{j-1} + r_k$. Thus, by the Second Gap Lemma, $x \le a < b$, a contradiction. If $\tilde{x} > b$, then $\tilde{x} \ge f^{(k)}_{j+1} > f^{(k)}_{j} + \varepsilon = b + \varepsilon$, a contradiction. Therefore, it must be $\tilde{x} = b$ and hence $x = b + \sum_{\substack{n\in A\\ n>k}} a_n \in b + E_k$. We have proved that

$$[b, b+\varepsilon] \cap E \;=\; [b, b+\varepsilon] \cap (b+E_k) \qquad \text{for } 0 < \varepsilon < f^{(k)}_{j+1} - f^{(k)}_{j}.$$

Finally, given $\varepsilon < \min\{a_k, f^{(k)}_{j+1} - f^{(k)}_{j}\}$, we get

$$\begin{aligned} b + \big([0,\varepsilon] \cap E\big) &\overset{\textbf{Obs. 1}}{=} b + \big([0,\varepsilon] \cap E_k\big) = [b, b+\varepsilon] \cap (b + E_k\big) \\ &\overset{\textbf{Obs. 2}}{=} [b, b+\varepsilon] \cap E. \end{aligned}$$

The proof of the second equality in the thesis of the Thm. 21.19 is analogous. $\square$

We are now ready for the main Guthrie-Nymann Classification Theorem (Thm. 1, [8]).

Theorem 21.20. *The set E of all subsums of an absolutely convergent series always is of one of the following four types:*
(i) a finite set;
(ii) a union of a finite family of bounded closed intervals;
(iii) a Cantor set;
(iv) an M-Cantorval.

Proof. Clearly, E is a finite set if and only if almost all terms of the series are zeros.

It remains to look at the case when $\sum a_n$ is a convergent series of positive terms and of sum 1. Assume that E is then neither a union of a finite family of closed intervals nor a Cantor set. The first assumption tells us that $a_n > r_n$ for infinitely many n by the Cor. 21.14. The second assumption tells us that E contains at least one closed interval by the Thm. 21.7.

Then 0 is the limit of a sequence of E-gaps by the First Gap Lemma. Since E is symmetric with respect to the point $\frac{1}{2}$, 1 is the limit of a sequence of E-gaps as well.

A union of a finite family of nowhere dense sets is nowhere dense. Hence, since E contains a component interval, it follows from the Fact 21.5 that sets E_k contain at least one component interval P_k. Since $E_n = [0, r_n] \cap E = [0, a_n) \cap E$ for all $n \in A := \{i : a_i > r_i\}$, it follows that the intervals P_k are intervals of E

for those n as well. The sequence of intervals $(P_n)_{n\in A}$ converges to 0, because $r_n \to 0$. By symmetry again, the point $1 \in E$ is the limit point of a sequence of E-intervals as well.

Now, by the Nymann-Sáenz Thm., we conclude that every endpoint of every E-gap is the limit of a sequence of E-gaps and of a sequence of E-intervals. Finally, an application of the Mendes-Oliveira Thm. 21.18 shows that E is an M-Cantorval. □

The latter theorem states that the space l_1 can be decomposed into four sets c_{00}, $\mathcal{C}$, $\mathcal{I}$ and $\mathcal{MC}$, where $\mathcal{I}$ consists of sequences (x_n) with $E(x_n)$ equal to a finite union of intervals, $\mathcal{C}$ consists of sequences (x_n) with $E(x_n)$ homeomorphic to the Cantor set, and $\mathcal{MC}$ consists of sequences (x_n) with $E(x_n)$ being Cantorvals. Let us recall some examples of absolutely summable sequences belonging to $\mathcal{MC}$. We use the original notations proposed by the authors. The notation will be unified later in the chapter.

A. D. Weinstein and B. E. Shapiro in [23] gave an example of a sequence (a_n) defined by the formulas: $a_{5n+1} = 0,24 \cdot 10^{-n}$, $a_{5n+2} = 0,21 \cdot 10^{-n}$, $a_{5n+3} = 0,18 \cdot 10^{-n}$, $a_{5n+4} = 0,15 \cdot 10^{-n}$, $a_{5n+5} = 0,12 \cdot 10^{-n}$. So,

$$(a_n) = \left(\frac{3\cdot 8}{10}, \frac{3\cdot 7}{10}, \frac{3\cdot 6}{10}, \frac{3\cdot 5}{10}, \frac{3\cdot 4}{10}, \frac{3\cdot 8}{100}, \dots \right).$$

However, they did not justify why the interior of $E(a_n)$ is non-empty.

Independently, C. Ferens ([7]) constructed a sequence (b_n) putting $b_{5l-m} = (m+3)\frac{2^{l-1}}{3^{3l}}$ for $m = 0,1,2,3,4$ and $l = 1,2,\dots$. Therefore

$$(b_n) = \left(7 \cdot \frac{1}{27}, 6 \cdot \frac{1}{27}, 5 \cdot \frac{1}{27}, 4 \cdot \frac{1}{27}, 3 \cdot \frac{1}{27}, 7 \cdot \frac{2}{27^2}, \dots \right).$$

Finally, in Jones' paper [10] there is presented a sequence

$$(d_n) = \left(\frac{3}{5}, \frac{2}{5}, \frac{2}{5}, \frac{2}{5}, \frac{3}{5} \cdot \frac{19}{109}, \frac{2}{5} \cdot \frac{19}{109}, \frac{2}{5} \cdot \frac{19}{109}, \frac{2}{5} \cdot \frac{19}{109}, \frac{3}{5} \cdot (\frac{19}{109})^2, \dots \right).$$

In fact, R. Jones shows a continuum of sequences generating Cantorvals, indexed by a parameter q, by proving that, for any positive number q with

$$\frac{1}{5} \leqslant \sum_{n=1}^{\infty} q^n < \frac{2}{9}$$

(i.e. $\frac{1}{6} \leqslant q < \frac{2}{11}$) the sequence

$$\left(\frac{3}{5},\frac{2}{5},\frac{2}{5},\frac{2}{5},\frac{3}{5}q,\frac{2}{5}q,\frac{2}{5}q,\frac{2}{5}q,\frac{3}{5}q^2,\dots\right)$$

is not in $\mathcal{C}$ nor $\mathcal{I}$, so it belongs to $\mathcal{MC}$. Based on Jones' idea, we will describe one-parameter families of sequences which contain (in particular) $(a_n),(b_n),(d_n)$ and many others.

For any $q \in (0,\frac{1}{2})$ we will use the symbol $(k_1,k_2,\dots,k_m;q)$ to denote the sequence $(k_1,k_2,\dots,k_m,k_1q,k_2q,\dots,k_mq,k_1q^2,k_2q^2,\dots,k_mq^2,\dots)$. Such sequences we will call multigeometric. In ([5]) the authors have obtained the following

Theorem 21.21. *Let $k_1 \geqslant k_2 \geqslant \dots \geqslant k_m$ be positive integers and $K = \sum_{i=1}^m k_i$. Assume that there exist positive integers n_0 and n such that each of numbers $n_0, n_0+1,\dots,n_0+n$ can be obtained by summing up the numbers $k_1,k_2,\dots,k_m$ (i.e. $n_0 + j = \sum_{i=1}^m \varepsilon_i k_i$ with $\varepsilon_i \in \{0,1\}, j = 1,\dots,n$). If*

$$\frac{1}{n+1} \leqslant q < \frac{k_m}{K+k_m}$$

then $E(k_1,\dots,k_m;q)$ is a Cantorval.

Now we can easily check that sequences $(a_n),(b_n)$ and (d_n) generate Cantorvals, because they belong to appropriate one-parameter families, indexed by q.

Example 21.22. The Weinstein-Shapiro sequence ([23]).

It is clear that if $E(x_n)$ is a Cantorval, $\alpha \neq 0$ and $(\alpha x_n) = (\alpha x_1, \alpha x_2, \dots)$, then $E(\alpha x_n)$ is a Cantorval too. To simplify a notation we multiply the sequence (a_n) by $\frac{10}{3}$ and consider the family of sequences

$$a_q = (8,7,6,5,4;q)$$

for $q \in (0,\frac{1}{2})$. Summing up $8,7,6,5$ and 4, we can get any natural number between $n_0 = 4$ and $n+n_0 = 26$. Therefore, by Theorem 21.21, for any q satisfying inequalities

$$\frac{1}{23} \leqslant q < \frac{4}{34},$$

the sequence a_q generates a Cantorval. Obviously, the number $\frac{1}{10}$ used in [23] belongs to $[\frac{1}{23},\frac{4}{34})$. It is not difficult to check that $a_q \in \mathcal{I}$ for $q \geqslant \frac{4}{34}$.

Example 21.23. The Ferens sequence ([7]).

For the family of sequences

$$b_q = (7,6,5,4,3;q)$$

K is equal to 25, $n_0 = 3$ and $n = 19$. Hence, for any $q \in [\frac{1}{20}, \frac{3}{28})$, b_q generates a Cantorval. In particular, the sequence $(7,6,5,4,3;\frac{2}{27})$, obtained from the Ferens sequence by multiplication by a constant, generates a Cantorval. Note that $b_q \in \mathcal{I}$, for $q \geqslant \frac{3}{28}$.

Example 21.24. The Jones-Velleman sequence ([10]).

Applying Theorem 21.21 to the sequence

$$d_q = (3,2,2,2;q)$$

we obtain $K = 9$, $n_0 = 2$ and $n = 5$, so for any $q \in [\frac{1}{6}, \frac{2}{11})$, $E(d_q)$ is a Cantorval set. Clearly, $\sum_{n=1}^{\infty} q^n \in [\frac{1}{5}, \frac{2}{9})$, for such q. Moreover $d_q \in \mathcal{I}$ for $q \geqslant \frac{2}{11}$.

We can also consider analogous sequences for more than three 2's. In fact, any sequence

$$x_q = (3,\underbrace{2,\ldots,2}_{k-times};q)$$

with $q \in [\frac{1}{2k}, \frac{2}{2k+5})$, generates a Cantorval set.

Note that for $k = 1$ and $k = 2$ the argument of Theorem 21.21 breaks down, because $\frac{1}{2k} > \frac{2}{2k+5}$.

However, we can apply Theorem 21.21 to "shortly defined" sequences. Indeed, for the sequence $(4,3,2;q)$, numbers K, n_0 and n are the same as for d_q. It is not difficult to check that, to keep the interval $[\frac{1}{n+1}, \frac{k_m}{K+k_m})$ non-empty, m should be greater than 2.

There is a natural question if Theorem 21.21 precisely describes the set of q with $(k_1,\ldots,k_m;q) \in \mathcal{MC}$. The upper bounds, for all mentioned examples are exact, because $(k_1,\ldots,k_m;q) \in \mathcal{I}$, for $q > \frac{k_m}{K+k_m}$. However, this is not true for all sequences satisfying the assumptions of Theorem 21.21.

Example 21.25. For the sequence $h_q = (10,9,8,7,6,5,2;q)$, we have $K = 47$, $n_0 = 5$ and $n = 37$. Therefore the interval $[\frac{1}{n+1}, \frac{k_m}{K+k_m}) = [\frac{1}{38}, \frac{2}{49})$ is non-empty.

However, for $h = (10,9,8,7,6,5,2;\frac{2}{49})$ and any $n \in \mathbb{N}$, we have $\sum_{i>7n-1} h(i) = (\frac{2}{49})^{n-1}(2 + \frac{\frac{2}{49}\cdot 47}{1-\frac{2}{49}}) = 4(\frac{2}{49})^{n-1} < h(7n-1)$. It means that $h \notin \mathcal{I}$. Note that in the second part of the proof of 21.21(compare [5]) only the inequality $q \geq \frac{1}{n+1}$ is used. Since $\frac{2}{49} > \frac{1}{38}$, we have $h \notin \mathcal{C}$ and so $h \in \mathcal{MC}$.

Again, it is not difficult to check that $h_q \notin \mathcal{I}$ if and only if $q < \frac{3}{50}$.

21.4 Topological and algebraic properties of $\mathcal{C}$, $\mathcal{I}$ and $\mathcal{MC}$

Let us observe that all the sets $c_{00}, \mathcal{C}, \mathcal{I}$ and $\mathcal{MC}$ are dense in ℓ_1. Moreover, c_{00} is an $\mathcal{F}_\sigma$-set of the first category. We are interested in studying the topological size and Borel classification of considered sets. To do it, let us consider the hyperspace $H(\mathbb{R})$, that is the space of all non-empty compact subsets of reals, equipped with the Vietoris topology (see [13], 4F, pp. 24-28). Recall, that the Vietoris topology is generated by the subbase of sets of the form $\{K \in H(\mathbb{R}) : K \subset U\}$ and $\{K \in H(\mathbb{R}) : K \cap U \neq \emptyset\}$ for all open sets U in $\mathbb{R}$. This topology is metrizable by the Hausdorff metric d_H given by the formula

$$d_H(A,B) = \max\{\max_{t\in A} d(t,B), \max_{s\in B} d(s,A)\}$$

where d is the natural metric in $\mathbb{R}$. It is known that the set N of all nowhere dense compact sets is a G_δ-set in $H(\mathbb{R})$ and the set F of all compact sets with finite number of connected components is an $\mathcal{F}_\sigma$-set. To see this, it is enough to observe that

- K is nowhere dense if and only if for any set U_n from a fixed countable base of natural topology in $\mathbb{R}$ there exists a set U_m from this base, such that $cl(U_m) \subset U_n$ and $K \subset (cl(U_m))^c$;
- K has more then k components if and only if there exist pairwise disjoint open intervals $J_1, J_2, \ldots, J_{k+1}$, such that $K \subset J_1 \cup J_2 \cup \cdots \cup J_{k+1}$ and $K \cap J_i \neq \emptyset$ for $i = 1, 2, \ldots, k+1$.

Now, let us observe that if we assign the set $E(x)$ to the sequence $x \in \ell_1$, we actually define the function $E : \ell_1 \to H(\mathbb{R})$.

It is not difficult to check (compare Lemma 3.1, [2]) that the function E is Lipschitz with Lipschitz constant $L = 1$, and consequently it is continuous. Now we can prove that

Theorem 21.26 ([2]). *The set $\mathcal{C}$ is a dense G_δ-set (and hence residual), $\mathcal{I}$ is a true $\mathcal{F}_\sigma$-set (i.e. it is $\mathcal{F}_\sigma$ but not $\mathcal{G}_\delta$) of the first category, and $\mathcal{MC}$ is in the class $(\mathcal{F}_{\sigma\delta} \cap \mathcal{G}_{\delta\sigma}) \setminus \mathcal{G}_\delta$.*

Proof. Let us observe that $\mathcal{C} \cup c_{00} = E^{-1}[N]$ and $\mathcal{I} \cup c_{00} = E^{-1}[F]$ where N, F, E are defined as before. Hence $\mathcal{C} \cup c_{00}$ is G_δ-set and $\mathcal{I} \cup c_{00}$ is $\mathcal{F}_\sigma$-set. Thus $\mathcal{C}$ is G_δ-set (because c_{00} is $\mathcal{F}_\sigma$-set) and $\mathcal{I} \cup \mathcal{MC}$ is $\mathcal{F}_\sigma$. Moreover, $\mathcal{I} = (\mathcal{I} \cup c_{00}) \cap (\mathcal{I} \cup \mathcal{MC})$ is $\mathcal{F}_\sigma$-set, too. By the density of $\mathcal{C}$, $\mathcal{C}$ is residual. Since $\mathcal{I}$ is dense of the first category, it cannot be $\mathcal{G}_\delta$-set. For the same reason, $\mathcal{MC}$ also cannot be $\mathcal{G}_\delta$-set. Since $\mathcal{MC}$ is a difference of two $\mathcal{F}_\sigma$-sets, it is in the class $\mathcal{F}_{\sigma\delta} \cap \mathcal{G}_{\delta\sigma}$. $\square$

Jones in a very nice paper [10] gives the following example. Let $(x_n) = (1/2^n)$ and $(y_n) = (1/3^n)$. Then clearly $(x_n) \in \mathcal{I}$ and $(y_n) \in \mathcal{C}$. Moreover, $(x_n + y_n) \in \mathcal{C}$ and $(x_n - y_n) \in \mathcal{I}$. Since, for any $n \in \mathbb{N}$, $x_n = (x_n + y_n) - y_n$ and $y_n = -(x_n - y_n) + x_n$, then neither $\mathcal{I}$ nor $\mathcal{C}$ is closed under pointwise addition. However, the sets $\mathcal{C}$, $\mathcal{I}$ and $\mathcal{MC}$ contain large ($\mathfrak{c}$-generated) algebraic structures.

Assume that V is a linear space (linear algebra). A subset $E \subset V$ is called lineable (algebrable) whenever $E \cup \{0\}$ contains an infinite-dimensional linear space (infinitely generated linear algebra, respectively). For a cardinal $\kappa > \omega$, let us observe that the set E is κ-algebrable (i.e. it contains κ-generated linear algebra), if and only if it contains an algebra which is a κ-dimensional linear space . Moreover, we say that a subset E of a commutative linear algebra V is strongly κ-algebrable, if there exists a κ-generated free algebra A contained in $E \cup \{0\}$. The subset M of a Banach space X is spaceable if $M \cup \{0\}$ contains infinitely dimensional closed subspace Y of X. (More information of such structures and a rich bibliography is presented in chapter 14.) In [2] it is proved that

Theorem 21.27. *$\mathcal{C}$ and $\mathcal{I}$ are strongly $\mathfrak{c}$-algebrable. $\mathcal{MC}$ is $\mathfrak{c}$-lineable.*

Theorem 21.28. *Let $\mathcal{I}_1$ be a subset of $\mathcal{I}$ which consists of those $x \in \ell_1$ for which $E(x)$ is an interval. Then $\mathcal{I}_1$ is spaceable. Moreover, for any infinite-dimensional closed subspace Y of ℓ_1, there is $(y_n) \in Y$ such that $E(y_n)$ contains an interval.*

Note that from the last assertion it follows that the set $\mathcal{C}$ - the biggest in the topological sense - is not spaceable.

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