Alexej P. Pynko

MINIMAL SEQUENT CALCULI FOR ŁUKASIEWICZ’S FINITELY-VALUED LOGICS*

Keywords: Sequent calculus, Łukasiewicz’s logics.

Abstract

The primary objective of this paper, which is an addendum to the author’s [8], is to apply the general study of the latter to Łukasiewicz’s n-valued logics [4]. The paper provides an analytical expression of a 2(n−1)-place sequent calculus (in the sense of [10, 9]) with the cut-elimination property and a strong completeness with respect to the logic involved which is most compact among similar calculi in the sense of a complexity of systems of premises of introduction rules. This together with a quite effective procedure of construction of an equality determinant (in the sense of [5]) for the logics involved to be extracted from the constructive proof of Proposition 6.10 of [6] yields an equally effective procedure of construction of both Gentzen-style [2] (i.e., 2-place) and Tait-style [11] (i.e., 1-place) minimal sequent calculi following the method of translations described in Subsection 4.2 of [7].

1. Introduction


*2010 Mathematics Subject Classification. Primary: 03B22, 03B50, 03F05; Secondary: 03F03.
The work is supported by the National Academy of Sciences of Ukraine.
studied in [8]. Łukasiewicz’s logics do deserve a particular emphasis because, as opposed to Dunn’s and Gödel’s logics, they do all have both equality determinant (in the sense of [5]) and singularity determinant (in the sense of [7])(cf. Proposition 6.10 of [6] and Corollary 6.2 of [7] for positive results as well as Propositions 6.5 and 6.8 therein for negative ones), in which case many-place sequent calculi (in the sense of [10, 9]) to be constructed following [8] for the former logics are naturally translated into both Gentzen-style [2](i.e., 2-place ) and Tait-style [11] (i.e., 1-place) sequent calculi according to Subsections 4.2.1 and 4.2.2 of [7].

2. Main results

$L = \{\neg, \land, \lor, \top\}$. Take any $n \geq 2$. Here we deal with the matrix underlying algebra $\mathfrak{A}_n$ specified as follows. The carrier $A_n$ of $\mathfrak{A}_n$ is set to be $n$. Finally, operations of $\mathfrak{A}_n$ are defined as follows:

\begin{align*}
-\mathfrak{A}_n a & \triangleq n - 1 - a, \\
 a \land \mathfrak{A}_n b & \triangleq \min(a, b), \\
 a \lor \mathfrak{A}_n b & \triangleq \max(a, b), \\
 a \supset \mathfrak{A}_n b & \triangleq \min(n - 1, n - 1 - a + b),
\end{align*}

for all $a, b \in A_n$.

**Lemma 2.1.** For any $i \in n \setminus \{0\}$ and any $j \in n \setminus \{n - 1\}$, we have the following introduction rules for $\mathcal{M}^{\mathfrak{A}_n}$:

\[
\left\{\{I_{n-1-i}:p_0\}\right\} \quad \left\{\{F_{n-1-j}:p_0\}\right\} \\
\{F_i:p_0\} \quad \{I_j:p_0\} \\
\{F_i:p_0, F_i:p_1\} \quad \{I_j:p_0, I_j:p_1\} \\
\{F_i:(p_0 \land p_1)\} \quad \{I_j:(p_0 \land p_1)\} \\
\{F_i:(p_0 \lor p_1)\} \quad \{I_j:(p_0 \lor p_1)\} \\
\{\{I_{n-2-k}:p_0, F_{i-k}:p_1\} \mid 0 \leq k < i\} \\
\{F_i:(p_0 \supset p_1)\} \\
\{\{F_{n-i}:p_0, I_{j-i}:p_1\} \mid 0 < l \leq j\} \cup \{\{F_{n-1-j}:p_0\}, \{I_j:p_1\}\} \\
\{I_j:(p_0 \supset p_1)\}
\]

\]
Proof: Let $i \in n \setminus \{0\}$ and $j \in n \setminus \{n - 1\}$. Checking (1) of [8] for the introduction rules of types $s: \gamma$, where $s \in \{F_i, I_j\}$ and $\gamma \in \{-, \land, \lor\}$, is trivial. As for those of types $s: \supset$, where $s \in \{F_i, I_j\}$, take any $a, b \in n$. Remark that $(a \supset A \land b) \in F_i \iff n - 1 - a + b \geq i$. Likewise, $(a \supset A \lor b) \in I_j \iff n - 1 - a + b \leq j$.

Suppose $n - 1 - a + b \geq i$, that is, $n - 1 - i + b \geq a$. Consider any $0 \leq k < i$. Suppose $a \in F_{n-1-k} = n \setminus I_{n-2-k}$, that is, $a \geq n - 1 - k$. Combining two inequalities, we get $k \geq i - b$, that is, $b \in F_{i-k}$.

Conversely, assume $n - 1 - a + b < i$, in which case $n - 1 - a < i$ too. As $0 \leq n - 1 - a$, we can choose $k \triangleq n - 1 - a$. If $a$ was in $I_{n-2-k}$, we would have $0 \leq -1$. Likewise, by the inequality under assumption, if $b$ was in $F_{i-k}$, we would have $b > b$. Thus, both $a \not\in I_{n-2-k}$ and $b \not\in F_{i-k}$.

Remark that (1) of [8] for the introduction rule of type $I_j$: $\supset$ is equivalent to the following condition:

$$n - 1 - a + b \leq j \iff \forall l \in (j + 2) : a \leq n - l - 1 \Rightarrow b \leq j - l \quad (2.1)$$

for all $a, b \in A_n$.

First, suppose $n - 1 - a + b \leq j$, that is, $n - 1 - j + b \leq a$. Consider any $l \in (j + 2)$. Assume $a \leq n - l - 1$. Combining two inequalities, we get $b \leq j - l$ as required.

Finally, assume $n - 1 - a + b > j$. Put $l \triangleq \min(n - 1 - a, j + 1)$. Then, $l \in (j + 2)$. Moreover, $a \leq n - l - 1$. If $b$ was not greater than $j - l$, we would have $l + b \leq j$, in which case $l \leq j$, and so $l = n - 1 - a$, in which case $n - 1 - a + b \leq j$. The contradiction with the inequality under assumption shows that $b > j - l$. Thus, (2.1) holds. This completes the argument. \qed

Notice that each of the sets of premises of rules involved in the formulation of Lemma 2.1 consists of functional $S_n$-signed $\emptyset$-sequents of some type $V \subseteq \text{Var}$ and forms an anti-chain with respect to $\preceq$. Then, by Theorem 2.15(ii) of [8], Lemma 2.1 yields
Theorem 2.2. For any $i \in n \setminus \{0\}$ and any $j \in n \setminus \{n - 1\}$:

\[
\begin{align*}
P_{F_i;\neg} & = \{\{I_{n-1-i};p_0\}\}, \\
P_{I_j;\neg} & = \{\{F_{n-1-j};p_0\}\}, \\
P_{F_i;\wedge} & = \{\{F_i;p_0\},\{F_i;p_1\}\}, \\
P_{I_j;\wedge} & = \{\{I_j;p_0\},\{I_j;p_1\}\}, \\
P_{F_i;\vee} & = \{\{F_i;p_0\},\{F_i;p_1\}\}, \\
P_{I_j;\vee} & = \{\{I_j;p_0\},\{I_j;p_1\}\}, \\
P_{F_i;\supset} & = \{\{I_{n-2-k};p_0, F_{i-k};p_1\} \mid 0 \leq k < i\}, \\
P_{I_j;\supset} & = \{\{F_{n-i-l};p_0, I_{j-l};p_1\} \mid 0 < l \leq j\} \cup \{\{F_{n-1-j};p_0\},\{I_j;p_1\}\}\end{align*}
\]

This provides the minimal $2(n-1)$-place sequent calculus for $\mathfrak{A}_n$. Notice that $P_{I_{n-2};\supset}$ has exactly $n$ elements. Remark that, in case $n = 2$, the resulted calculus coincides with Gentzen’s classical calculus $LK$ [2].

References


Department of Digital Automata Theory (100)
V. M. Glushkov Institute of Cybernetics
Academician Glushkov prosp. 40
Kiev, 03680, Ukraine
e-mail: pynko@voliacable.com