PORTFOLIO CONSTRUCTION WITH MODIFIED SHARPE’S METHOD

ABSTRACT. In the paper we consider a modification of Sharpe’s method used in classical portfolio analysis for optimal portfolio building.

The key idea of the paper is the modification of the classical approach by application of the errors-in-variable model. We assume that both independent (market portfolio return) as well as dependent (given asset’s return) variables are randomly distributed values related with each other by linear relationship and we build the model used for parameters’ estimation.

For model evaluation we made a comparison of portfolios comprising nine stocks from Warsaw Stock Exchange, which are built using classical Sharpe’s and proposed method.

Key words: Sharpe’s model, errors-in-variable model, estimation, comparison.

I. CLASSIC SHARPE’S MODEL

In classic Sharpe’s model return of \( k \)-th asset’s \( R^k \) is explained by market portfolio return \( R^m \) through characteristic line equation according to the relationship:

\[
R^k = \beta_k (R^m - r_f) + r_f + \epsilon_k,
\]

where \( r_f \) is riskless rate of interest and \( \epsilon_k \) has normal distribution. Knowing the market sensitivity \( \beta_k \) of \( k \)-th asset we receive the formula for the expected value and the variance of portfolio return ratio \( R_p \):

\[
E(R_p) = \beta (E(R^m) - r_f) + r_f, \quad D^2(R_p) = \beta^2 D^2(R^m) + \sum_{k=1}^{K} x_k^2 D^2(\epsilon_k),
\]

\( \beta, x_k \in \mathbb{R} \), \( \epsilon_k \sim \text{Normal}(0,1) \)

\( \beta \) is market sensitivity of \( k \)-th asset.
where \( \beta = \sum_{k=1}^{K} x_k \beta_k \); \( x_k \) - \( k \)-th stock’s share in investment portfolio. The problem of optimal portfolio choice may be reduced to finding \( \max \left[ E(R_p) - \lambda D^2(R_p) \right] \) with condition: \( x_k \geq 0, \sum_{k=1}^{K} x_k = 1 \), where \( \lambda \) - risk/profit exchange ratio coefficient. In this model the \( \beta_k \) coefficient is estimated by least square method.

II. MODIFIED SHARPE’S MODEL

Let us consider the situation, in which both: dependent variable, being the certain asset’s surplus return ratio and denoted \( R^k \) as well as independent variable - the market portfolio surplus return ratio \( R^m \) are disturbed observations related with each other by linear equation, i.e.:

\[
R^m - r_f = \xi + \varepsilon; \quad R^k - r_f = \beta_k \xi + \alpha_k + \delta_k,
\]

If we assume that random variables \( \varepsilon, \delta_k \) have normal distribution with unknown parameters then model (2) is unidentifiable. The proof was given by Reiersol(1950).

In that case, replication of measurement of each pair of observation: dependent and independent; \( m \) times overcomes the nonidentifiability (see Bunke & Bunke(1989)).

The following approach was therefore proposed: returns of given asset and market portfolio are analysed within periods of \( t \). To be more specific the period of one month has been assumed as \( t \).

Let \( n \) denotes number of historical months, \( m \) number of monthly returns during given month and \( K \) number of assets in portfolio. Let \( R^m_{ij} \) and \( R^k_{ij} \) are \( j \)-th monthly return ratios in the \( i \)-th month of portfolio market and \( k \)-th asset, respectively and \( X_{ij} = R^m_{ij} - r_f, Y^k_{ij} = R^k_{ij} - r_f \).

We consider the model:

\[
X_{ij} = \xi_{ij} + \varepsilon_{ij}, \quad Y^k_{ij} = \beta_k \xi_{ij} + \alpha_k + \delta^k_{ij}, \quad i = 1, \ldots, n, j = 1, \ldots, m, k = 1, \ldots, K,
\]

It has been assumed that \( \delta_{ij} \sim N(0, \sigma^2_{\delta_{ij}}), \xi_{ij} \sim N(s_i, \sigma^2_{\xi}), \varepsilon_{ij} \sim N(0, \sigma^2_{\varepsilon}) \).

All the parameters of aforementioned distributions are unknown.
Now the issue of optimal portfolio choice is finding: \( \max \left[ E(R_p) - \lambda D^2(R_p) \right] \)
under conditions: \( x_k \geq 0, \sum_{k=1}^{K} x_k = 1 \) with:

\[
E(R_p) = \sum_{k=1}^{K} x_k (\beta_k s + r_f), \quad D^2(R_p) = (\sum_{k=1}^{K} x_k \beta_k)^2 \sigma_s^2 + \sum_{k=1}^{K} x_k^2 \sigma_{\delta_k}^2. \quad (4)
\]

where \( s \) is the expected value for distribution of monthly returns of market portfolio during the period of the prognosis.

III. ESTIMATION OF UNKNOWN PARAMETERS

First step – Estimation of parameters for given asset

For given \( k \) in model (3) the number of unknown parameters to be estimated increases with \( n \). Unknown parameters are: \( \beta_k, \alpha_k, \sigma_{\delta_k}^2, \sigma_s^2, \sigma_{\epsilon}^2, s_1, \ldots, s_n \). It can be shown, that in this model the maximum likelihood estimators have normal distribution asymptotically with respect to \( m \) and \( n \) (see Dolby(1976)).

Let:

\[
X_i = \frac{1}{m} \sum_{j=1}^{m} X_{ij}, \quad Y_i = \frac{1}{m} \sum_{j=1}^{m} Y_{ij}, \quad X_\cdot = \frac{1}{n} \sum_{i=1}^{n} X_i, \quad Y_\cdot = \frac{1}{n} \sum_{i=1}^{n} Y_i,
\]

and subsequently:

\[
w_{xx} = \frac{1}{mn} \sum_{i=1}^{n} \sum_{j=1}^{m} (X_{ij} - X_i)^2, \quad w_{yy} = \frac{1}{mn} \sum_{i=1}^{n} \sum_{j=1}^{m} (Y_{ij} - Y_i)^2,
\]
\[
w_{xy} = \frac{1}{mn} \sum_{i=1}^{n} \sum_{j=1}^{m} (X_{ij} - X_i)(Y_{ij} - Y_i),
\]
\[
s_{xx} = \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{m} (X_{ij} - X_\cdot)^2, \quad s_{yy} = \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{m} (Y_{ij} - Y_\cdot)^2,
\]
\[
s_{xy} = \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{m} (X_{ij} - X_\cdot)(Y_{ij} - Y_\cdot),
\]

the maximum likelihood estimator of \( \beta_k \) and \( \alpha_k \) coefficients have forms:
\[ \hat{\beta}_k = \frac{w_{xy}^k s_{xx} - w_{xx} s_{yy}^k - \sqrt{\Delta}}{2(w_{xy}^k s_{xx} - w_{xx} s_{xy})} \quad \hat{a}_k = Y_{ij}^k - \hat{\beta}_k X_{ij}, \]

where \( \Delta = (w_{xx} s_{yy}^k - s_{xx} w_{yy}^k)^2 - 4(s_{xy}^k w_{yy}^k - s_{yy} w_{xy}^k)(s_{xx} w_{xy}^k - s_{xy} w_{xx}^k). \)

The estimators of all the other values are expressed in terms of \( \hat{\beta}_k \) (Cox(1976)). These estimators have forms of:

\[ \hat{\sigma}_{sk}^2 = \frac{w_{xy}^k}{\hat{\beta}_k} - p_k q_k B(\hat{\beta}_k)/\hat{\beta}_k \quad \hat{\sigma}_{sk}^2 = q_k (W(\hat{\beta}_k) + B(\hat{\beta}_k)) \]

where, \( B(\hat{\beta}_k) = s_{yy} - 2\hat{\beta}_k s_{xy} + \hat{\beta}_k^2 s_{xx}, \quad W(\hat{\beta}_k) = w_{xy}^k - 2\hat{\beta}_k w_{xy}^k + \hat{\beta}_k^2 w_{xx} \)

and \( p_k = (\hat{\beta}_k w_{xx} - w_{xy}^k)/W(\hat{\beta}_k), \quad q_k = (w_{xy}^k - \hat{\beta}_k w_{xy}^k)/W(\hat{\beta}_k). \)

**Second step - Estimation of unknown parameters for the whole model**

For given \( j \) we construct vectors: \( x_j^T, y_j^{kT} \) as:

\[ x_j^T = (X_j^1, ..., X_{nj}), \quad y_j^{kT} = (Y_{k1j}, ..., Y_{knj}), \quad k = 1...K. \]

The random variables \( \mu_j = (x_j^T, y_j^{1T}, y_j^{2T}, ..., y_j^{kT}) \) are independent with the same normal distribution and the vector of expected values \( \mu = (s^T, (\beta_s + \alpha_s)^T, ..., (\beta_K s + \alpha_K)^T) \) where: \( s^T = (s_1, ..., s_n) \) and with covariance matrix \( V \):

\[
V = \begin{bmatrix}
\sigma_s^2 + \sigma_{e}^2 & \beta_1 \sigma_s^2 & \cdots & \beta_K \sigma_s^2 \\
\beta_1 \sigma_s^2 & \beta_1^2 \sigma_s^2 + \sigma_{e_1}^2 & \cdots & \beta_1 \beta_K \sigma_s^2 \\
\vdots & \vdots & \ddots & \vdots \\
\beta_K \sigma_s^2 & \beta_1 \beta_K \sigma_s^2 & \cdots & \beta_K^2 \sigma_s^2 + \sigma_{e_K}^2
\end{bmatrix} \otimes I_n = \sum \otimes I_n.
\]

The logarithm of likehood function for variable \( \mu_j \) with unknown parameters of: \( \beta_1, ..., \beta_K, \alpha_1, ..., \alpha_K, \sigma_s^2, \sigma_{e}^2, \sigma_{e_1}^2, ..., \sigma_{e_K}^2, s_1, ..., s_n \), has a form of:
\[ \log L(\mu_j, \Psi) = C - \frac{m}{2} \log(\det(V)) - \frac{1}{2} \sum_{j=1}^{m} d_j V^{-1} d_j, \]

where \( d_j = (\mu_j - \mu) \) and \( \Psi \) is the set of unknown parameters.

Let \( V_\psi \) denote a matrix composed of matrix \( V \) elements after derivation with respect to \( \psi \) parameters (\( \psi \) being the arbitrary element of \( \Psi \) set). The \( V \) matrix is symmetric thus:

\[
\frac{\partial}{\partial \psi} \log L(\mu_j, \Psi) = m \left\{ \frac{1}{2} \text{tr}(PV_\psi) - d_\psi^T V^{-1} d \right\}, \text{ with } P = V^{-1} (D - V)V^{-1}
\]

and \( D = \frac{1}{m} \sum_{j=1}^{m} d_j d_j^T \).

It comes that: \( d_\psi^T V^{-1} d = 0 \), \( \frac{1}{2} \text{tr}(PV_{\beta_k}) - d_{\beta_k}^T V^{-1} d = 0 \), \( d_{\sigma_k}^T V^{-1} d = 0 \), \( \text{tr}(PV_{\sigma_k^2}) = 0 \), \( \text{tr}(PV_{\sigma_k^2}) = 0 \), \( \text{tr}(PV_{\delta_k^2}) = 0 \).

From equations \( d_\psi^T V^{-1} d = 0 \), \( d_{\sigma_k}^T V^{-1} d = 0 \) we received relationships:

\[ \alpha_k = Y_k - \beta_k X \text{ and } s_i = \frac{X_i + A_i (Y^1 - Y^1 + \beta_1 X_1) + \ldots + A_k (Y^K - Y^K + \beta_K X)}{1 + A_i \beta_1 + \ldots + A_k \beta_k} \]  

(5)

where: \( A_k = \beta_k \sigma_k^2 / \sigma_{\delta_k}^2 \).

From: \( \text{tr}(PV_{\sigma_k^2}) = 0 \), \( \text{tr}(PV_{\sigma_k^2}) = 0 \), exploiting certain features of matrix algebra we obtain:

\[ \sigma_r^2 + \sigma_s^2 = \sum_{i=1}^{n} \sum_{j=1}^{m} \frac{(X_{ij} - s_i)^2}{mn} \sigma_{\delta_k}^2 + \beta_k^2 \sigma_s^2 = \sum_{i=1}^{n} \sum_{j=1}^{m} \frac{(Y_{ij} - \beta_k s_i - \alpha_k)^2}{mn}. \]  

(6)

From equations (5), (6) and relationship \( \text{tr}(PV_{\sigma_k^2}) = 0 \) we calculated \( \sigma_r^2 \):

\[ \sigma_r^2 = \sum_{k=1}^{K} \left( 1 - \frac{T_k (\beta_k)}{\sigma_{\delta_k}^2} \right) \left/ \left( \frac{1}{2} \sum_{i=1}^{K} \sum_{k=1}^{K} \frac{\sigma_{\delta_k}^2}{\sigma_{\delta_k}^2 \sigma_{\delta_i}^2} - K \sum_{k=1}^{K} \frac{\beta_k^2}{\sigma_{\delta_k}^2} \right) \right. \]  

(7)
with:

\[ T^k(\beta_k) = (s_{xy}^k + w_{xy}^k) - 2 \beta_k (s_{xy}^k + w_{xy}^k) + \beta_k^2 (s_{xx} + w_{xx}), \]

\[ T^{kl}(\beta_k, \beta_l) = \beta_k^2 (s_{xy}^{kl} + w_{xy}^{kl}) - 2 \beta_k \beta_l (s_{xy}^{kl} + w_{xy}^{kl}) + \beta_l^2 (s_{xy}^{kl} + w_{xy}^{kl}), \]

\[ w_{xy}^{kl} = \frac{1}{nm} \sum_{i=1}^{n} \sum_{j=1}^{m} (Y_{ij}^k - Y_i^k)(Y_{ij}^l - Y_i^l), \]

\[ s_{xy}^{kl} = \frac{1}{n} \sum_{i=1}^{n} (Y_i^k - Y_i^l)(Y_i^l - Y_i). \]

From relationships (6) eliminating \( \sigma^2_s \) parameters, we received \( k \) of square equations with \( \sigma^2_e \) as variable. Applying the Viète’s formulas for each of them we received the \( \sigma^2_{\delta_k} \) in terms of \( \sigma^2_{\delta_l} \) for \( k=2, \ldots, K \):

\[ \sigma^2_{\delta_k} = \frac{\beta_k^2}{\beta_l^2} \left( \sigma^2_{\delta_l} - T^l(\beta_l) - 2 \beta_l w_{xy}^l \right) + T^k(\beta_k) + 2 \beta_k w_{xy}^k. \]  

(8)

Substituting \( \sigma^2_{\delta_k} \) calculated according to (8) into equation (7), we received relationship between \( \sigma^2_e \) and \( \sigma^2_{\delta_l} \). The parameter \( \sigma^2_e \) obtained in such a way we substitute into one of \( k \) square equations, which gives a basis for calculating \( \sigma^2_{\delta_l} \).

Ultimately the maximum of likelihood function depends only on unknown \( \beta_1, \ldots, \beta_K \). The last may be determined numerically starting from initial values equal to their estimators calculated in the first step.

**IV. AN EXAMPLE PORTFOLIO**

An example portfolio consists of nine companies listed on the Warsaw Stock Exchange (debica, KGHM, krosno, orbis, PKNorlen, sokolow, TPSA, woleczanka, zywiec - to be more specific). As the market portfolio the portfolio being the base of WIG index was assumed. The analysis was performed on archival data starting from January 2000 until March 2006.

In practice in portfolio construction it is sufficient to take estimators calculated in the first step. Slight differences in \( \sigma^2_s \) estimated for various assets have been
corrected by averaging. For estimation of unknown value of $s_i$, $\sum_{i=1}^{n} s_i$ was assumed. It can be shown that: $\sum_{i=1}^{n} s_i = x$.

To illustrate the features of portfolio constructed with use of modified method we compared it with that constructed using classical method. It appears that both methods construct quite different contents. Moreover for lambda values, for which there is no significant further portfolio diversification classical method "prefers" (in decreasing order): zywiec, krosno, debica, PKNorlen, sokolow; while the modified one: PKNorlen, zywiec, TPSA, sokolow. For illustration the expected value and the variance of the portfolio being rebuilt during 14 months have been presented.

Figures 1 and 2 present expected return ratio and its variance for both methods, respectively. It turned out that the expected value is greater than resulting from analysis of the classic model and the variance is lower. To depict the difference in constructing models we present the real monthly return ratio for optimal contents for both portfolios for chosen $\lambda$ coefficient. The verification was done using past market price (exchange ratio) of the asset in the month, for which the prognosis was made. Picture 3 contains the comparison of real profits for both discussed methods. The WIG return $R_m$ is also included as the background reference. It may be noticed that low-risk portfolio built using the proposed method gives per average profits greater than that of classic Sharpe’s portfolio.

![Fig. 1. Comparison of variance for both methods](image-url)
Fig. 2. Expected return ratio for both methods

Fig. 3. Real profit comparison for both methods.

V. CONCLUSIONS

A new method of portfolio construction has been proposed. The main assumptions is that, the variable explaining the assets’ return ratios is a random variable biased by a random error with normal distribution. The method of estimators’ construction in errors-in-variable model extended for many variables
has been shown. Comparing parameters of portfolios constructed with classic Sharpe’s method and the proposed one, we found that expected returns for our method are bigger, while its variance is lower. In terms of real profits it means that for higher $\lambda$ (portfolios’ diversification is practically “saturated”) are greater for the presented model.

REFERENCES

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**BUDOWA PORTFELA AKCJI ZA POMOCĄ ZMODYFIKOWANEJ METODY SHARPE’A**


Przyjęto, że zarówno stopa zwrotu danego waloru jak i stopa zwrotu portfela rynkowego są pewnymi zaburzonymi już wartościami, między którymi istnieje zależność liniowa.
Dla ilustracji zagadnienia porównano portfele składające się z dziewięciu spółek zbudowane w oparciu o klasyczną metodę Sharpe’a i proponowaną jej modyfikację. Jako portfel rynkowy przyjęto portfel leżący u podstaw indeksu giełdowego WIG. Analizę przeprowadzono na podstawie notowań archiwalnych od stycznia 2000 do marca 2006. Budując wyżej wymienione portfele miesięczne dokonano porównań przebudowując je co miesiąc uzyskując w ten sposób wektor składów do analizy porównawczej.