GOLDFELD-QUANDT TEST: UNBIASEDNESS VS SYMMETRY

ABSTRACT. The Goldfeld-Quandt test for homoscedasticity in a classical linear regression appears to be biased. In the paper an unbiased test is constructed. The result is extended to families of distributions with scale parameter.

Key words: Goldfeld-Quandt test, test for scale, unbiased test, chi-square test for variance.

I. INTRODUCTION

Consider a classical linear regression problem \( Y_i = \beta_0 + \beta_1 x_i + \varepsilon_i, \ i = 1, \ldots, n \). It is assumed that random errors \( \varepsilon_i \)'s are independent random variables distributed as \( N(0, \sigma^2) \). After fitting the model the homoscedasticity of random errors should be checked. One of the commonly applied tests is the Goldfeld-Quandt test (Greene 2000). In the test, the sample is divided into two disjoint subsamples of size \( n_1 \) and \( n_2 \) respectively, so two models are considered \( Y_i = \beta_0^{(1)} + \beta_1^{(1)} x_i + \varepsilon_i^{(1)}, \ i = 1, \ldots, n_1 \) and \( Y_i^{(2)} = \beta_0^{(2)} + \beta_1^{(2)} x_i + \varepsilon_i^{(2)}, \ i = 1, \ldots, n_2 \) with the assumption: \( \varepsilon_i^{(1)} \sim N(0, \sigma_1^2) \) and \( \varepsilon_i^{(2)} \sim N(0, \sigma_2^2) \). Verified hypothesis is \( H_0: \sigma_1^2 = \sigma_2^2 \) vs \( H_1: \sigma_1^2 \neq \sigma_2^2 \). The Goldfeld-Quandt test statistic is \( S_1^2 / S_2^2 \), where \( S_1^2 \) and \( S_2^2 \) are residual variances in the first and the second model respectively. Hypothesis is rejected at the significance level \( \alpha \), if

\[
\frac{S_1^2}{S_2^2} < F\left(1 - \frac{\alpha}{2}, n_1 - 2, n_2 - 2\right) \quad \text{or} \quad \frac{S_2^2}{S_1^2} > F\left(\frac{\alpha}{2}, n_1 - 2, n_2 - 2\right),
\]

where \( F(\alpha; u, v) \) is the \( \alpha \)-critical value of the F distribution with \( (u, v) \) degrees of freedom. Unfortunately, it appears that the test is biased one, i.e. the power of the test may be smaller than the significance level. In the Figure 1 the power of the test for \( n_1 = 10, n_2 = 15 \) and \( \alpha = 0.05 \) is shown.

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Hence it is harder to reject the hypothesis $H_0: \sigma_1^2 = \sigma_2^2$ for some variances $\sigma_1^2$ smaller than $\sigma_2^2$. In the following table there are given variances from the alternative hypothesis along with appropriate probabilities $p$ of rejecting $H_0$:

<table>
<thead>
<tr>
<th>$\frac{\sigma_1^2}{\sigma_2^2}$</th>
<th>0.896</th>
<th>0.912</th>
<th>0.928</th>
<th>0.944</th>
<th>0.960</th>
<th>0.976</th>
<th>0.992</th>
<th>1.000</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p$</td>
<td>0.05004</td>
<td>0.04947</td>
<td>0.04913</td>
<td>0.04898</td>
<td>0.04904</td>
<td>0.04929</td>
<td>0.04972</td>
<td>0.05000</td>
</tr>
</tbody>
</table>

In practice application of the test may lead to serious misstatement that variances are equal though they are not.

The question is, does there exist an unbiased version of Goldfeld-Quandt test and, if yes, how to construct such a test. The above test will be referred to as a classical one.

The similar situation is met in the problem of testing $H_0: \sigma^2 = \sigma_0^2$ vs $H_1: \sigma^2 \neq \sigma_0^2$ in normal distribution $N(\mu, \sigma^2)$. The most commonly used test (e.g. Bickel and Doksum 1980; Bartoszyński and Niewiadomska-Bugaj 1996, Chow 1983, Müller 1991, Storm 1979), based on a sample $X_1, ..., X_n$ with the sample mean $\bar{X} = (X_1 + ... + X_n)/n$, is as follows: reject $H_0$ at a significance level $\alpha$ if

$$\frac{\sum_{i=1}^n (X_i - \bar{X})^2}{\sigma_0^2} < \chi^2\left(1 - \frac{\alpha}{2}; n-1\right) \quad \text{or} \quad \frac{\sum_{i=1}^n (X_i - \bar{X})^2}{\sigma_0^2} > \chi^2\left(\frac{\alpha}{2}; n-1\right).$$
Here $\chi^2(\alpha; \nu)$ denotes the $\alpha$-critical value of the chi-square distribution with $\nu$ degrees of freedom. Some statistical packages (for example Statgraphics and Statistica) implement exactly the above lower and upper critical values of the test. An unbiased test for this hypothesis may be found in Lehmann (1986). A construction of an unbiased test for the scale parameter of the exponential distribution and $n = 1$ one can find in Knight (2000). In what follows we give a general construction of an unbiased test for scale families of distributions. Some important for practical applications examples as well as a remark on the shortest confidence intervals, are also presented.

II. UNBIASED TEST FOR SCALE PARAMETER

Consider a statistical model $\left( R_+, \{ F_\lambda(x), \lambda > 0 \} \right)$ with scale parameter, i.e. $F_\lambda(x) = F(x/\lambda)$. The problem is to test $H_0 : \lambda = \lambda_0$ vs $H_1 : \lambda \neq \lambda_0$ at a significance level $\alpha$. If $T$ is a scale equivariant estimator of $\lambda$, then critical region is of the form:

\[ (*) \quad \frac{T}{\lambda_0} < G^{-1}(\alpha_2) \quad \text{or} \quad \frac{T}{\lambda_0} > G^{-1}(\alpha_1), \]

where $\alpha_1 + \alpha_2 = \alpha$ and $G^{-1}$ is the quantile function of the null distribution of $T/\lambda_0$.

**Theorem.** There exist $\alpha_1, \alpha_2 > 0$ such that $\alpha_1 + \alpha_2 = \alpha$ and the test with critical region (*) is unbiased.

**Proof.** The power of the test for a fixed $\alpha_1 \in (0, \alpha)$ is

\[
M(\lambda) = P_\lambda \left\{ \frac{T}{\lambda_0} < G^{-1}(\alpha - \alpha_1) \quad \text{or} \quad \frac{T}{\lambda_0} > G^{-1}(1 - \alpha_1) \right\} = \\
= P_\lambda \left\{ \frac{T}{\lambda} < \frac{\lambda_0}{\lambda} G^{-1}(\alpha - \alpha_1) \quad \text{or} \quad \frac{T}{\lambda} > \frac{\lambda_0}{\lambda} G^{-1}(1 - \alpha_1) \right\} = \\
= G\left( \frac{\lambda_0}{\lambda} G^{-1}(\alpha - \alpha_1) \right) + \left( 1 - G\left( \frac{\lambda_0}{\lambda} G^{-1}(1 - \alpha_1) \right) \right). 
\]
Then
\[
\frac{dM(\lambda)}{d\lambda} = -\frac{\lambda_0 G^{-1}(\alpha - \alpha_1)}{\lambda^2} g\left(\frac{\lambda_0}{\lambda} G^{-1}(\alpha - \alpha_1)\right) + \frac{\lambda_0 G^{-1}(1 - \alpha_1)}{\lambda^2} g\left(\frac{\lambda_0}{\lambda} G^{-1}(1 - \alpha_1)\right).
\]

It follows that $M(\lambda)$ is strictly decreasing to the left of an $\lambda_*$, strictly increasing to the right of $\lambda_*$, and achieves its minimum at $\lambda_*$ such that
\[
G^{-1}(\alpha - \alpha_1) g\left(\frac{\lambda_0}{\lambda_*} G^{-1}(\alpha - \alpha_1)\right) = G^{-1}(1 - \alpha_1) g\left(\frac{\lambda_0}{\lambda_*} G^{-1}(1 - \alpha_1)\right).
\]

The test is unbiased iff $\lambda_* = \lambda_0$ which holds iff $0 < \alpha_1 < \alpha$ is a solution of the equation
\[
(*) \quad G^{-1}(\alpha - \alpha_1) g(G^{-1}(\alpha - \alpha_1)) = G^{-1}(1 - \alpha_1) g(G^{-1}(1 - \alpha_1))
\]

Note that if $\alpha_1 \to 0$ then LHS of $(*)$ tends to $G^{-1}(\alpha) g\left(G^{-1}(\alpha)\right) > 0$ and RHS of $(*)$ tends to zero. On the other hand, if $\alpha_1 \to 0$ then LHS of $(*)$ tends to zero and RHS of $(*)$ tends to $G^{-1}(1 - \alpha) g\left(G^{-1}(1 - \alpha)\right) > 0$. Hence, there exists $\alpha_1$ which is a solution of $(*)$. □

Critical values of an unbiased test are illustrated in the Fig. 2.

![Figure 2. Critical values of the unbiased test](image)
It is obvious that the power of the unbiased test and the power of the symmetric test are not comparable. Those powers are shown in the Fig. 3.

![Graph showing powers of unbiased and symmetric tests](image)

**III. THREE IMPORTANT EXAMPLES**

1. Let $F$ be the exponential distribution, i.e. $F_\theta(x) = 1 - \exp\left\{-\frac{x}{\theta}\right\}$ and consider the problem of testing $H_0 : \theta = \theta_0$ vs $H_1 : \theta \neq \theta_0$. Let $X_1, \ldots, X_n$ be a sample and let $T = \sum_{i=1}^{n} X_i$. The distribution of $T$ is the gamma one with density function:

$$g_\theta(t) = \frac{1}{\theta^n \Gamma(n)} t^{n-1} \exp\left\{-\frac{t}{\theta}\right\}, \quad \text{for } t > 0.$$ 

To find the unbiased test one has to find $\alpha_1$ such that

$$G_{1}^{-1}(\alpha - \alpha_1) g_1\left(G_{1}^{-1}(\alpha - \alpha_1)\right) = G_{1}^{-1}(1 - \alpha_1) g_1\left(G_{1}^{-1}(1 - \alpha_1)\right).$$

The solution may be find with the aid of computer (for $n = 1$ see Knight 2000). In the Table 1 (for $\alpha = 0.05$) there are given critical values for unbiased test ($u_1$ and $u_2$) as well as for classical test ($c_1$ and $c_2$).
2. Let $F$ be the normal distribution $N(\mu, \sigma^2)$ and consider the problem of testing $H_0: \sigma^2 = \sigma_0^2$ vs $H_1: \sigma^2 \neq \sigma_0^2$. Let $X_1, ..., X_n$ be a sample and let $T = \sum_{i=1}^{n} (X_i - \bar{X})^2$, where $\bar{X}$ denotes sample mean. The distribution of $T$ is the chi-square with $n - 1$ degrees of freedom. To find the unbiased test one has to find $a$ such that

$$G^{-1}(\alpha - \alpha_1)g(G^{-1}(\alpha - \alpha_1)) = G^{-1}(1 - \alpha_1)g(G^{-1}(1 - \alpha_1)),$$

where $G$ and $g$ denotes cdf and pdf of chi-square distribution with $n - 1$ degrees of freedom respectively. The solution may be find with the aid of computer. In the Table 2 (for $\alpha = 0.05$) there are given critical values for unbiased test ($u_1$ and $u_2$) as well as for classical test ($c_1$ and $c_2$).

Table 1

Critical values of unbiased and classical test in exponential distribution

<table>
<thead>
<tr>
<th>$n$</th>
<th>$\alpha - \alpha_1$</th>
<th>$u_1$</th>
<th>$u_2$</th>
<th>$c_1$</th>
<th>$c_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>0.0189</td>
<td>4.979</td>
<td>17.613</td>
<td>4.795</td>
<td>17.085</td>
</tr>
<tr>
<td>20</td>
<td>0.0207</td>
<td>12.439</td>
<td>30.137</td>
<td>12.217</td>
<td>29.671</td>
</tr>
<tr>
<td>50</td>
<td>0.0223</td>
<td>37.372</td>
<td>65.195</td>
<td>37.111</td>
<td>64.781</td>
</tr>
<tr>
<td>100</td>
<td>0.0231</td>
<td>81.645</td>
<td>120.919</td>
<td>81.364</td>
<td>120.529</td>
</tr>
</tbody>
</table>

Table 2

Critical values of unbiased and classical test in normal

<table>
<thead>
<tr>
<th>$n$</th>
<th>$\alpha - \alpha_1$</th>
<th>$u_1$</th>
<th>$u_2$</th>
<th>$c_1$</th>
<th>$c_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>0.0161</td>
<td>2.953</td>
<td>20.305</td>
<td>2.700</td>
<td>19.023</td>
</tr>
<tr>
<td>20</td>
<td>0.0188</td>
<td>9.267</td>
<td>33.921</td>
<td>8.907</td>
<td>32.852</td>
</tr>
<tr>
<td>50</td>
<td>0.0211</td>
<td>32.020</td>
<td>71.128</td>
<td>31.555</td>
<td>70.222</td>
</tr>
<tr>
<td>100</td>
<td>0.0222</td>
<td>73.882</td>
<td>129.253</td>
<td>73.361</td>
<td>128.422</td>
</tr>
</tbody>
</table>

3. Consider the problem of testing $H_0: \sigma_1^2 = \sigma_2^2$ vs $H_1: \sigma_1^2 \neq \sigma_2^2$ in two normal distributions with variances $\sigma_1^2$ and $\sigma_2^2$. Note that the Goldfeld-Quandt test is a special case of the considered test. Let $S_1^2$ and $S_2^2$ be two estimators respec-
tively of $\sigma_1^2$ and $\sigma_2^2$ such that they are independent random variables distributed as $\chi^2(v_1)/v_1$ and $\chi^2(v_2)/v_2$ and let $T = S_1^2/S_2^2$. The distribution of $T$ is the $F$ (Snedecor) with $v_1$ and $v_2$ degrees of freedom. In the Goldfeld-Quandt test $v_1 = n_1 - 2$ and $v_2 = n_2 - 2$. To find the unbiased test one has to find $\alpha_1$ such that

$$G_{v_1,v_2}^{-1}(\alpha - \alpha_1)g_{v_1,v_2}\left(G_{v_1,v_2}^{-1}(\alpha - \alpha_1)\right) = G_{v_1,v_2}^{-1}(1 - \alpha_1)g_{v_1,v_2}\left(G_{v_1,v_2}^{-1}(1 - \alpha_1)\right),$$

where $G_{v_1,v_2}$ and $g_{v_1,v_2}$ denotes cdf and pdf of $F$ (Snedecor) distribution with $v_1$ and $v_2$ degrees of freedom respectively. The solution may be find with the aid of computer. In the Table 3 (for $\alpha = 0.05$) there are given critical values for unbiased test ($u_1$ and $u_2$) as well as for classical test ($c_1$ and $c_2$).

<table>
<thead>
<tr>
<th>$n$</th>
<th>$m$</th>
<th>$\alpha - \alpha_1$</th>
<th>$u_1$</th>
<th>$u_2$</th>
<th>$c_1$</th>
<th>$c_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>10</td>
<td>0.0250</td>
<td>0.269</td>
<td>3.717</td>
<td>0.269</td>
<td>3.717</td>
</tr>
<tr>
<td>20</td>
<td>10</td>
<td>0.0215</td>
<td>0.348</td>
<td>3.290</td>
<td>0.361</td>
<td>3.419</td>
</tr>
<tr>
<td>50</td>
<td>10</td>
<td>0.0188</td>
<td>0.410</td>
<td>3.026</td>
<td>0.432</td>
<td>3.221</td>
</tr>
<tr>
<td>100</td>
<td>10</td>
<td>0.0177</td>
<td>0.434</td>
<td>2.936</td>
<td>0.459</td>
<td>3.152</td>
</tr>
</tbody>
</table>

Note that if $v_1 = v_2$ then $g_{v_1,v_1}(x) = g_{v_1,v_1}\left(\frac{1}{x}\right)$. Hence, in the case the unbiased test is the classical one, i.e. $\alpha_1 = \alpha/2$.

IV. UNBIASED TEST AND THE SHORTEST CONFIDENCE INTERVAL

Now consider the problem of constructing the shortest confidence interval for scale parameter $\lambda$ at the confidence level $1 - \alpha$. Because $T$ is scale equivariant estimator of $\lambda$ then the confidence interval for $\lambda$ is of the form:

$$\lambda \in \left(\frac{T}{G^{-1}(1 - \alpha_1)}; \frac{T}{G^{-1}(\alpha - \alpha_1)}\right).$$

(\text{**})
The shortest confidence interval is the solution of the following problem:

\[
\frac{1}{G^{-1}(\alpha - \alpha_1)} - \frac{1}{G^{-1}(1 - \alpha_1)} = \min, \\
\int_{G^{-1}(\alpha - \alpha_1)}^{G^{-1}(1 - \alpha_1)} g(x)dx = \alpha.
\]

Application of the method of Lagrange multipliers gives the following condition for quantiles of the distribution of \(T\) statistics:

\[
G^{-1}(\alpha - \alpha_1)g\left(G^{-1}(\alpha - \alpha_1)\right) = G^{-1}(1 - \alpha_1)g\left(G^{-1}(1 - \alpha_1)\right).
\]

This is the same condition as obtained for unbiased test. Hence, the shortest confidence interval for \(\lambda\) is the acceptance region in the unbiased test for \(H_0: \lambda = \lambda_0\) vs \(H_1: \lambda \neq \lambda_0\).

V. NUMERICAL IMPLEMENTATION

Critical values of an unbiased test may be found numerically. In what follows there is given a short Mathematica program which allows to find critical values of the unbiased version of the Goldfeld-Quandt test. For other problems similar programs may be written. Of course, there is also a possibility to use other mathematical or statistical packages (in a similar way) to find out critical values of an unbiased test.

```math
<<Statistics\'ContinuousDistributions\'
G[n_, m_] = FRatioDistribution[n, m] (*definition of a F distribution*)
Kw[n_, m_, q_] := Quantile[G[n, m], q] (*quantile function*)
FF[n_, m_, x_] := CDF[G[n, m], x] (*cumulative distribution function*)
HH[n_, m_, x_] := PDF[G[n, m], x] (*probability density function*)
RR[n_, m_, alfa_, beta_] := Kw[n, m, 1 - (alfa - beta)]*HH[n, m, Kw[n, m, 1 - (alfa - beta)]] - Kw[n, m, beta]*HH[n, m, Kw[n, m, beta]] (*equation to be solved with respect to beta=alfa-alfa1*)
alfa = 0.05; n = 20; m = 10;
b1 = beta /. FindRoot[RR[n, m, alfa, beta] == 0, beta, 0.01]
u1 = Kw[n, m, b1]
u2 = Kw[n, m, 1 - (alfa - b1)]
```
VI. CONCLUSIONS

It was shown that classical test for scale parameter is biased. It is recom­mended to use unbiased test critical values of which are nowadays easy obtain­able by standard software. An additional advantage is such that the shortest con­fidence intervals for scale parameters may be constructed.

REFERENCES


Storm R. (1979), Wahrscheinlichkeitsrechnung, mathematische Statistik und statistische Qualitätskontrolle, Veb Fachbuchverlag Leipzig (p. 147).

Statistical package: Statgraphics.

Statistical package: Statistica.

Wojciech Zieliński

TEST GOLDFELDA-QUANDTA: NIEOBCIĄŻONOŚĆ A SYMETRIA

Test Goldfelda-Quandta homoscedastyczności stosowany w klasycznym modelu regresji liniowej jest testem obciążonym. W pracy skonstruowano odpowiedni test nie­obciążony. Wynik został rozszerzony na rodziny rozkładów z parametrem skali.