Henryka Siejka

SOME EXTREMAL PROBLEMS IN THE CLASS
OF HOLOMORPHIC UNIVALENT FUNCTIONS

Let $S'(b), 0 < b < 1$, denote the class of functions of the form

$$ F(z) = z + \sum_{n=2}^{\infty} a_n z^n, \text{ holomorphic and univalent in the disc } |z| < 1, \text{ satisfying the condition } |F(z)| < b^{-1} $$

and let $G(\omega) = \sum_{k=1}^{\infty} c_k \omega^k$ denote the function inverse to $F$.

In the paper the estimation of some initial coefficients of $G$ as well as the estimation of $a_5$ in $S'(b)$ and in the odd subclass of $S'(b)$ are given for some $b$ from the interval $(0, 1)$.

1. INTRODUCTION

Let $S(b), 0 < b < 1$, denote the class of functions of the form

$$ f(z) = b(z + \sum_{n=2}^{\infty} a_n z^n) $$

holomorphic and univalent in the disc $D = \{z: |z| < 1\}$ and satisfying the condition $|f(z)| < 1$.

Denote by $S'(b), 0 < b < 1$, the class of holomorphic-univalent functions of the form

$$ \omega = F(z) = z + \sum_{n=3}^{\infty} a_n z^n, \quad z \in D, $$

satisfying the condition $|F(z)| < b^{-1}$ and let

$$ z = G(\omega) = \sum_{k=1}^{\infty} c_k \omega^k $$

denote the function inverse to $F$. 

[123]
Recently many authors (e.g. [3], [4], [9]) considered the problem of coefficient estimations in some classes of functions inverse to classes of meromorphic functions. In [1] Launonen estimated coefficients of the inverse functions of \( S(b) \) applying Fitz-Gerald-Launonen inequality.

In this paper the estimation of some initial coefficients of functions inverse to \( S'(b) \) functions will be considered. The Launonen method and the Power-inequality will be used.

2. THE ESTIMATION OF THE COEFFICIENTS \( c_3, c_4, c_5, c_7, c_9 \)

It follows from the connection between the functions \( f, F \) and \( G \) that
\[
\begin{align*}
    c_1 &= 1, \\
    c_2 &= 0, \\
    c_3 &= -a_3, \\
    c_4 &= -a_4, \\
    c_5 &= -a_5 + 3a_3^2
\end{align*}
\]

The estimation of \( c_3 \) and \( c_4 \) follows then immediately from the estimation of \( a_3 \) and \( a_4 \) in the class \( S(b) \) in the case \( a_2 = 0 \).

From [7] (p. 265) we have
\[
|c_3| = |a_3| \leq 1 - b^2
\]  
(3)
and the equality holds for the function \( G_o \) which is inverse to the one satisfying the equation
\[
\frac{F_o}{1 + b^2 F_o^2} = \frac{z}{1 + z^2}
\]
(4)

From Grunsky-type inequalities which are sharp in the considered case \( a_2 = 0 \) ([5]) it follows that
\[
\text{re } a_4 \leq \frac{2}{3} (1 - b^3) - \frac{|a_3|^2}{2(1 - b)} \leq \frac{2}{3} (1 - b^3).
\]
(5)
The equality holds for \( a_3 = a_2 = 0 \) and the extremal function \( F_1 \) is defined by
\[
\frac{F_1}{(1 - b^3 F_1^3)^{2/3}} = \frac{z}{(1 - z^2)^{2/3}}
\]
Thus
\[ |c_4| \leq \frac{2}{3} (1 - b^3) \]
with the equality for the function inverse to \( F_1 \).

In order to estimate next coefficients we apply the Launonen inequality [1]. The inequality for every function \( z = G(\omega) \) inverse to \( S'(b) \)-function has the form
\[
\left| \int_Y \mu(\omega) \frac{G(\omega) - G(\sigma)}{\omega - \sigma} \, d\omega \right| \leq \int_Y \mu(\omega) \frac{G(\omega) G(\sigma)}{\omega - \sigma} \frac{1 - b^2 \omega \sigma}{1 - G(\omega) G(\sigma)} \, d\omega \, d\sigma, \quad (6)
\]
where \( \gamma \) is a closed analytic curve and \( \mu \) is a continuous weight function on \( \gamma \). For \( \mu(\omega) = \omega^{-3} \) (6) takes the form
\[
|-4\pi^2 c_5| \leq 4\pi^2 (|c_3|^2 + 1 - b^2).
\]
Thus by (3) we have
\[
|c_5| \leq |c_3|^2 + 1 - b^2 \leq 2 - 3b^2 + b^4 \quad (7)
\]
The maximum is reached by the same function as in the case \( |c_3| \) i.e. by the function inverse to \( F_0 \).

For \( \mu(\omega) = \omega^{-4} \) the condition (6) yields
\[
|-4\pi^2 c_7| \leq 4\pi^2 (|c_4|^2 + |c_3|^2 (4 - b^2) + 1 - b^2),
\]
from where
\[
|c_7| \leq |c_4|^2 + |c_3|^2 (4 - b^2) + 1 - b^2 \quad (8)
\]
In order to estimate this we apply the area inequality for the class \( S(b) \) ([7], p. 182):
\[
\sum_{\nu=1}^{\infty} \nu |a_\nu - b^2 a_\nu|^2 \leq 1 \quad (9)
\]
where
\[
\frac{1}{F(z)} = \frac{1}{z} + \sum_{\nu=0}^{\infty} a_\nu z^\nu \quad (10)
\]
From (1) and (10) it follows that
\[
a_1 = -a_3, \quad a_2 = -a_4,
\]
and as the consequence of (9) we have
\[ |a_3 - b^2|^2 + 2|a_4|^2 \leq 1, \]

and by (2)

\[ |c_4|^2 \leq \frac{1}{2} (1 - |c_3 - b^2|^2) \quad (11) \]

The inequality (8) takes then the form

\[ |c_7| \leq \frac{1}{2} - \frac{1}{2} |c_3 - b^2|^2 + |c_3|^2 (4 - b^2) + 1 - b^2. \]

The rotated function \( \tau^{-1} G(\tau \omega) = \omega + \sum_{k=3}^{\infty} \tau^k c_k \omega^k, \quad ||\tau|| = 1, \)
preserves \( |c_7| \) and allows the normalization \( c_3 \leq 0 \). Denoting

\[ x = c_3 \in -(1 - b^2); 0> \] and \( P(x) = \left(\frac{7}{2} - b^2\right)x^2 + b^2x, \)

we have then

\[ |c_7| \leq \frac{3}{2} - b^2 - \frac{1}{2} b^4 + P(x). \]

Require

\[ P(-(1 - b^2)) = (1 - b^2) (b^4 - \frac{11}{2} b^2 + \frac{7}{2}) \geq P(0) = 0. \]

This yields

\[ \max P = P(-(1 - b^2)) \quad \text{for} \quad 0 \leq b \leq b_0, \]

where

\[ b_0 = \frac{1}{2} (11 - \sqrt{65})^{1/2} \approx 0.856992160... \quad (12) \]

is the root of the equation \( b^4 - \frac{11}{2} b^2 + \frac{7}{2} = 0. \) Hence for \( 0 \leq b \leq b_0 \) the sharp estimation holds

\[ |c_7| \leq \frac{3}{2} - b^2 - \frac{1}{2} b^4 + P(-(1 - b^2)) = 5 - 10b^2 + 6b^4 - b^6. \]

As in the case \( |c_5| \) the coefficient \( |c_7| \) is maximized with \( |c_3| \) i.e. by the function inverse to \( F_0. \)

In the case \( \mu(\omega) = \omega^{-5} \) we can proceed similarly. From (6) we have

\[ |c_9| \leq (9 - 4b^2)|c_3|^2 + (4 - b^2)|c_4|^2 + |c_5|^2 + 1 - b^2, \]

from where by (7), (11) and the fact that again we can assume \( c_3 \leq 0, \) there holds the inequality

\[ |c_9| \leq \frac{1}{2} (1 - b^2)(8 + 3b^2 - 2b^3 - b^4) + xQ(x), \]

where

\[ x = c_3 \in -(1 - b^2); 0> \] and
and
\[ Q(x) = x^3 + x(9 - \frac{11}{2}b^2) + b^2(4 - b^2). \]
Since \( Q'(x) > 0 \) for \( b \in (0, 1) \) and \( x \in \langle-(1 - b^2), 0\rangle \) then the equation \( Q(x) = 0 \) can have only one root in the interval \( \langle-(1 - b^2), 0\rangle \).

Require
\[ Q(-(1 - b^2)) = -(1 - b^2)(-b^6 - \frac{19}{2}b^4 + 43b^2 - 10) \geq 0. \]
This holds for \( 0 < b \leq b_1 \), where \( b_1 = 0.843285210... \) is the root of the equation
\[ -b^6 - \frac{19}{2}b^4 + \frac{43}{2}b^2 - 10 = 0 \quad (13) \]
Hence for \( 0 < b \leq b_1 \) the sharp estimation
\[ |c_9| \leq 14 - 35b^2 + 30b^4 - 10b^6 + b^8 \]
holds. Again with \( |c_3| \) also \( |c_9| \) is maximized by the function inverse to \( F_0 \).

So we have shown
THEOREM 1. For every function
\[ z = G(\omega) = \sum_{c=1}^{\infty} c_k \omega^k, \]
inverse to \( S'(b) \)-functions, the estimations
\[ |c_3| \leq 1 - b^2, \quad b \in (0, 1), \]
\[ |c_4| \leq \frac{2}{3}(1 - b^3), \quad b \in (0, 1), \]
\[ |c_5| \leq 2 - 3b^2 + b^4, \quad b \in (0, 1), \]
\[ |c_7| \leq 5 - 10b^2 + 6b^4 - b^6, \quad b \in (0, b_0), \]
\[ |c_9| \leq 14 - 35b^2 + 30b^4 - 10b^6 + b^8, \quad b \in (0, b_1) \]
hold, where \( b_0 \) is given by (12) and \( b_1 \) is the root of equation (13). Except for \( |c_4| \) the function inverse to \( F_0 \) defined by (4) is the extremal one. In the case \( |c_4| \) the extremal function is the one inverse to \( F_1 \) given by (5).

3. ON THE ESTIMATION OF \( a_5 \) IN THE CLASS \( S'(b) \)

From the Grunsky-type inequality for \( a_5 \) for which \( a_2 = 0 \) (see [2], p. 473) we have
where we denoted $a_3 = u + iv$. Provided $3 - \frac{2}{\ln b^{-1}} \leq 0$ what implies two cases

\begin{align*}
a) & \quad 1 - \frac{2}{3 \ln b^{-1}} < 0 \iff e^{-\frac{2}{3}} < b < 1, \\
b) & \quad 1 - \frac{2}{3 \ln b^{-1}} = 0 \iff b = e^{-\frac{2}{3}}.
\end{align*}

we obtain

$$\text{re } a_5 \leq \frac{1}{2} (1 - b^4).$$

In the case a) the equality in (14) holds for $u = v = a_3 = 0$. In the case b) it requires $v = 0$ but $u$ is left as a free parameter.

As the rotation $\tau^{-1}F(\tau z)$ preserves the class $S'(b)$ we can assume that $a_5 = |a_5| \geq 0$ and $\text{re } a_3 \leq 0$.

In order to study the equality cases in (15) put $a_2 = 0$ in the inequality (82), p. 472 of [2]. We obtain

$$\text{re } (\ln bx_0^2 + a_3 x_1^2 + a_5 - \frac{3}{2} a_3^2 + 2a_3 x_0 + 2a_4 x_1) \leq (1 - b^2)|x_1|^2 + \frac{1}{2} (1 - b^4),$$

where $x_0$, $x_1$ are free complex parameters. Since, in the normalized equality case of (14), $a_5 = \frac{1}{2} (1 - b^4)$, $v = 0$, the above inequality takes the form

$$\text{re } (\ln bx_0^2 + ux_1^2 - \frac{3}{2} u^2 + 2ux_0 + 2a_4 x_1) \leq (1 - b^2)|x_1|^2.$$  \hspace{1cm} (16)

Putting $x_0 = 0$, $x_1 = |x_1|e^{i\phi}$ in the case (a) $u = 0$ we have

$$2 \text{re } (e^{i\phi}a_4) \leq (1 - b^2)|x_1|^2$$

what with $0 < |x_1| + 0$ gives $\text{re } (e^{i\phi}a_4) \leq 0$ for $\phi \in <0, 2\pi>$ which implies $a_4 = 0$.

In order to prove that also in the case (b) $a_4 = 0$ it is sufficient to put $x_0 = \frac{3}{2} u$, $x_1 = |x_1|e^{i\phi}$ in the inequality (16) and tends with $|x_1|$ to zero.
So we have shown that in the extremal case all the coefficients up to $a_5$ are real. From the Power inequality it then follows that we may use condition (35), p. 488 in [6]:

$$2x_o \ln bF + b^2(b^2F^2 - b^{-2}F^{-2}) = 2x_o \ln z + z^2 - z^{-2},$$

$$2x_o = a_3 = u \leq 0$$  

(17)

In the case (a) in (16) $u = x_o = 0$. In the case (b) the extremal case can be studied by aid of the boundary correspondence. For that purpose let us put in (17) $z = e^{i\phi}$, $F(e^{i\phi}) = r(\phi)e^{i\psi(\phi)}$ and compare the real parts:

$$u \ln br + e^{-\frac{4}{3}} \cos 2\psi(b^2r^2 - b^{-2}r^{-2}) = 0,$$

from where

$$\cos 2\psi 0 - e^{4/3} \frac{u \ln br}{b^2r^2 - b^{-2}r^{-2}} \frac{-ue^{4/3}}{r + b^{-1}}$$

what implies the limitation for $u$:

$$-4e^{-\frac{4}{3}} \leq u \leq 0.$$

So we have proved

**Theorem 2.** In the class $S'(b)$ for $b \in (e^{-\frac{2}{3}}, 1)$ the estimation

$$|a_5| \leq \frac{1}{2} (1 - b^4)$$

holds. The equality holds for the function given by (17) where $x_o = 0$ for $b \in (e^{-\frac{2}{3}}, 1)$ and arbitrary $x_o = u \in (-4e^{-\frac{4}{3}}, 0)$ for $b = e^{-\frac{2}{3}}$.

4. ON THE ESTIMATION OF $a_5$ IN THE ODD SUBCLASS OF $S'(b)$

The problem of estimation $a_5$ for $b \in (0, e^{-\frac{2}{3}})$ remains open in the class $S'(b)$ but we solve the corresponding question in the odd subclass of $S'(b)$. We will use the well-known fact that if $f(z)$ of form (1) belongs to $S(b)$ then
\( \tilde{f}(z) = \sqrt{f(z^2)} = \tilde{a}(z + \tilde{a} z^3 + \ldots), \)

is an odd function from \( S(b^{1/2}) \) and the connections

\[
\begin{align*}
    b &= \tilde{b}^2, \\
    a_2 &= 2\tilde{a}_3, \\
    a_3 &= 2\tilde{a}_5 + \tilde{a}_3^2
\end{align*}
\]

hold.

From the Power inequality in the class \( S(b) \), [8] we have

\[
\text{re} \ (a_3 - a_2^2) \leq 1 - b^2 + \frac{u^2}{\ln b}, \quad \text{where} \quad u = \text{re} \ a_2,
\]

and the equality can be reached if \( |U| \leq 2b|\ln b| \). In the terms of odd \( S(\tilde{b}) \) functions it gives

\[
2 \text{re} \ \tilde{a}_5 - (1 - \tilde{b}^4) \leq 3 \text{re} \ \tilde{a}_2^2 + 4 \frac{(\text{re} \ \tilde{a}_3)^2}{\ln \tilde{b}^2} =
\]

\[
= (3 + \frac{4}{\ln \tilde{b}^2})\tilde{u}^2 - 3\tilde{v}^2 \leq (3 + \frac{4}{\ln \tilde{b}^2})\tilde{u}^2 = \tilde{M}(\tilde{u}),
\]

where we denote \( \tilde{a}_3 = \tilde{u} + i\tilde{v} \). Equality holds for \( \tilde{v} = 0 \), and if \( \tilde{b} \in (e^{-\frac{2}{3}}, 1) \) we obtain former estimation

\[
\text{re} \ \tilde{a}_5 \leq \frac{1}{2} (1 - \tilde{b}^4).
\]

If \( \tilde{b} \in (0, e^{-\frac{2}{3}}) \) and \( |\tilde{u}| \leq \tilde{b}^2 |\ln \tilde{b}^2| \) we have an estimation

\[
\tilde{M}(\tilde{u}) \leq (3 + \frac{4}{\ln \tilde{b}^2})\tilde{b}^4 |\ln \tilde{b}^2| \leq \tilde{b}^4 |\ln \tilde{b}^2| (4 + 3 |\ln \tilde{b}^2|) \quad (18)
\]

For \( |\tilde{u}| \geq \tilde{b}^2 |\ln \tilde{b}^2| \) what is equivalent to \( |\text{re} \ a_2| = |U| \geq 2b|\ln b| \) from [8], p. 17 we obtain

\[
\text{re} \ (a_3 - a_2^2) \leq 1 - b^2 - 2|U|\sigma + 2(\sigma - b)^2,
\]

where \( \sigma \in <b, 1> \) is the root of the equation

\[
\sigma \ln \sigma - \sigma + b + \frac{|U|}{2} = 0.
\]

In terms of \( S'(\tilde{b}) \) it means that

\[
2 \text{re} \ \tilde{a}_5 - (1 - \tilde{b}^4) \leq 3(\tilde{u}^2 - \tilde{v}^2) - 4|\tilde{u}|\tilde{\sigma}^2 + 2(\tilde{\sigma}^2 - \tilde{b}^2)^2;
\]

\[
\tilde{\sigma}^2 = \sigma \in <\tilde{b}^2, 1>,
\]

\[
|\tilde{u}| = - (\tilde{\sigma}^2 \ln \tilde{\sigma}^2 - \tilde{\sigma}^2 + \tilde{b}^2).
\]
So, for \( \Re z_5 \) \[ \ln |z|^2 \leq |u| \leq 1 - \frac{\alpha^2}{2}, \]
\[ 2 \Re z_5 - (1 - \alpha^4) \leq 3u^2 - 4|u|\alpha^2 + 2(\omega^2 - \alpha^2)^2, \]
\[ |u| = -(\omega^2 \ln \omega^2 - \omega^2 + \beta^2), \]
with the equality for \( \alpha = 0 \). To estimate the upper bound, return for brevity to the variable \( \sigma \):
\[ 2 \Re z_5 - (1 - \beta^4) \leq 3(\sigma \ln \sigma + \sigma + b)^2 + 4\sigma(\sigma \ln \sigma + \sigma + b) + \]
\[ + 2(\sigma - b)^2 = M(\sigma), \quad \beta^2 = b \leq \sigma \leq 1. \] (19)

Since \( \sigma = \sigma(u) \in (b, 1) \) is uniquely determined by
\[ |u| \in (\beta^2 |\ln \beta^2|, \; 1 - \beta^2), \]
([8], p. 15), then it is sufficient to maximize \( M(\sigma) \) for \( \sigma \in (b, 1) \). Since
\[ \frac{dM(\sigma)}{d\sigma} = 2 \ln \sigma(3\sigma \ln \sigma + \sigma + 3b) \]
the considered maximum is reached for the root of the equation
\[ 3\sigma \ln \sigma + \sigma + 3b = 0 \]
which belongs to the interval \( e^{-\frac{4}{3}}, 1 \rangle \), what by (19) gives
\[ \Re z_5 \leq \frac{1}{2} (1 - \beta^4) + (\omega^2 - \beta^2)^2. \]
Since the maximum of \( \hat{M}(\hat{u}) \) in (18) equals to \( M(\sigma) \) we have maximized \( \Re z_5 \) and hence \( |z_5| \):

**THEOREM 3.** In the odd subclass of \( S'(b), \; b \in (0, e^{-\frac{2}{3}}) \),
\[ |a_5| \leq \frac{1}{2} (1 - \beta^4) + (\omega^2 - \beta^2), \]
where \( \sigma \in (e^{-\frac{2}{3}}, 1) \) is the root of the equation
\[ 3\sigma^2 \ln \sigma^2 + \sigma^2 + 3b^2 = 0. \]

**REFERENCES**


Niech $S(\beta)$, $0 < \beta < 1$, oznacza klasę funkcji postaci
\[ \omega = F(z) = z + \sum_{n=3}^{\infty} a_n z^n, \]
holomorficznych i jednolistnych w kręgu $|z| < 1$, spełniających tam $|F(z)| < b^{-1}$ i niech funkcja
\[ z = G(\omega) = \sum_{k=1}^{\infty} c_k \omega^k \]
będzie funkcją odwrotną od $F$.

W przedstawionym artykule otrzymano oszacowania współczynników $c_3$, $c_4$, $c_5$, $c_7$, $c_9$ dla pewnych $\beta$ z przedziału $(0, 1)$. Ponadto otrzymano oszacowanie współczynnika $a_5$ w klasie $S(\beta)$ dla $\beta < e^{-2/3}$ oraz w podklasie funkcji nieparzystych tej klasy dla $\beta < (0, e^{-2/3})$. Stosowano metodę Launonen oraz pewne nierówności potęgowele (Power-inequality).