For a typical continuous function $f$ on $[0,1]$, $f$ has an I-essential derived number at each point $x \in (0,1)$.

Zajíček proved [3] that, for a typical continuous real-valued function $f$ and each $x \in (0,1)$, there exists $y \in \mathbb{R}$ which is an essential derived number of $f$ at $x$. In this paper we shall prove that this theorem remains true if we replace the notion of an essential derived number of $f$ at $x$ by an analogous notion for the Baire category.

Let $C$ denote the set of continuous real valued functions defined on $[0,1]$ furnished with the metric of uniform convergence. When we say a typical $f \in C$ has a certain property $\mathcal{P}$, we shall mean that the set of $f \in C$ with this property is residual in $C$.

The notation used throughout this paper is standard. In particular, $\mathbb{R}$ stands for the set of real numbers, $\mathbb{R} = \mathbb{R} \cup \{-\infty, \infty\}$, $I$ for the $\sigma$-ideal of sets of the first category, $\|f\|$ for the norm in $C$, $B(f,r)$ for the open ball in $C$ with centre $f$ and radius $r$ and $\chi_A$ for the characteristic function of a set $A$.

**Definition 1.** ([1]) We say that $x_0 \in \mathbb{R}$ is an upper $I$-density point of a set $A$ having the Baire property if and only if there exists an increasing sequence of real numbers $\{t_n\}_{n \in \mathbb{N}}$ tending to infinity, such
that

\[ \chi_{t_n(E-x_0) \cap (-1,1)} \rightarrow 1 \text{ with respect to } I \text{ as } n \to \infty \text{ on } (-1,1) \]

(see [2] for the definition of the convergence with respect to \( I \)). We shall use the notation \( d_I(E, x_0) = 1 \).

Observe that \( x_0 \) is an upper \( I \)-density point of a set \( A \) if and only if \( 0 \) is an upper \( I \)-density point of \( A - x_0 = \{x - x_0 : x \in A\} \).

It is easy to see that \( d_I(E, x_0) = 1 \) if and only if there exists an increasing sequence of real numbers \( \{t_n\}_{n \in \mathbb{N}} \) tending to infinity, such that

\[ \lim_{n \to \infty} \chi_{t_n(E-x_0) \cap (-1,1)}(x) = 1 \]

\( I \)-a.e. on \((-1,1)\).

**Definition 2.** We say that \( y \) is an \( I \)-essential derived number of \( f \) at \( x \) if there exists a set \( E \subset \mathcal{R} \) having the Baire property, such that \( d_I(E, x) = 1 \) and \( \lim_{t \to x, t \in E} \frac{f(t) - f(x)}{t - x} = y \).

**Theorem.** For a typical \( f \in \mathcal{C} \) and each \( x \in (0,1) \), there exists \( y \in \mathcal{R} \) which is an \( I \)-essential derived number of \( f \) at \( x \).

**Proof.** Let \( \{P_k\}_{k \in \mathbb{N}} \) be a sequence of polynomials which is dense in \( \mathcal{C} \). For each \( k \in \mathbb{N} \), put \( M_k = \|P_k''\| = \sup_{x \in [0,1]} |P_k''(x)| \) and choose \( \delta_k \) such that \( 0 < \delta_k < (kM_k)^{-1} \) and \( \delta_k \downarrow 0 \) as \( k \to \infty \). Let

\[ G = \bigcap_{m=1}^{\infty} \bigcup_{k=m}^{\infty} B\left(P_k, \frac{\delta_k}{4k^2}\right) = \limsup_k B\left(P_k, \frac{\delta_k}{4k^2}\right). \]

For each \( m \in \mathbb{N} \), the set \( \bigcup_{k=m}^{\infty} B(P_k, \frac{\delta_k}{4k^2}) \) is open and dense, so \( G \) is a dense \( G_\delta \)-subset of \( \mathcal{C} \). Hence \( G \) is residual in \( \mathcal{C} \). Choose an arbitrary \( f \in G \). It is sufficient to prove that, for each \( x \in (0,1) \), there exists an \( I \)-essential derived number of \( f \) at \( x \). Fix \( x_0 \in (0,1) \). Since \( f \in G \), we can choose an increasing sequence of positive integers \( \{k_n\}_{n \in \mathbb{N}} \) such that \( f \in B(P_{k_n}, \delta_{k_n} \cdot (4k_n^2)^{-1}) \) for each \( n \in \mathbb{N} \). Let \( h_n = \delta_{k_n} \), \( A_n = P_k(x_0) \), \( z_n = (k_n)^{-1} \). Since \( \delta_n \downarrow 0 \) as \( n \to \infty \), we have \( h_n \downarrow 0 \) for \( n \to \infty \). For \( n \) large enough, we get \((x_0 - h_n, x_0 + h_n) \subset (0,1) \). We
shall show that, for such an $n$ and for $x \in (x_0 - h_n, x_0 - h_n(k_n)^{-1}) \cup (x_0 + h_n(k_n)^{-1}, x_0 + h_n)$, we have

$$\left| \frac{f(x) - f(x_0)}{x - x_0} - A_n \right| < z_n.$$ 

We can assume that $x \in (x_0 + \frac{h_n}{k_n}, x_0 + h_n)$ (the other case is analogous). We have

$$\left| \frac{f(x) - f(x_0)}{x - x_0} - \frac{P_{kn}(x) - P_{kn}(x_0)}{x - x_0} \right| < \left| \frac{f(x) - P_{kn}(x)}{x - x_0} \right| + \left| \frac{f(x_0) - P_{kn}(x_0)}{x - x_0} \right|$$

(1)

$$< \frac{2\delta_{kn}(4k_n^2)^{-1}}{\delta_{kn}(k_n)^{-1}} = \frac{1}{2k_n}.$$

By the Taylor formula, for some $\xi \in (0, 1)$,

$$\left| \frac{P_{kn}(x) - P_{kn}(x_0)}{x - x_0} - P_{kn}'(x_0) \right| = \left| \frac{1}{2} P_{kn}''(\xi)(x - x_0) \right|$$

(2)

$$< \frac{1}{2} M_{kn} \delta_{kn} < \frac{1}{2} M_{kn} \cdot \frac{1}{k_n M_{kn}}$$

$$= \frac{1}{2k_n}.$$

From (1) and (2) we obtain

$$\left| \frac{f(x) - f(x_0)}{x - x_0} - P_{kn}'(x_0) \right| < \frac{1}{k_n},$$

hence

(3)

$$\left| \frac{f(x) - f(x_0)}{x - x_0} - A_n \right| < z_n$$

for $x \in (x_0 - h_n, x_0 - h_n(k_n)^{-1}) \cup (x_0 + h_n(k_n)^{-1}, x_0 + h_n)$. 
Denote by \( y \in \mathbb{R} \) a cluster point of a sequence \( \{A_n\}_{n \in \mathbb{N}} \). Then there exists a subsequence \( \{n_p\}_{p \in \mathbb{N}} \) of the sequence of positive integers, such that \( A_{n_p} \to y \) as \( p \to \infty \). Define

\[
E = \bigcup_{p=1}^{\infty} \left[ \left( x_0 - \frac{h_{n_p}}{k_{n_p}}, x_0 - \frac{h_{n_p}}{k_{n_p}} \right) \cup \left( x_0 + \frac{h_{n_p}}{k_{n_p}}, x_0 + h_{n_p} \right) \right].
\]

We shall show that

\[
\bar{d}_f(E, x_0) = 1 \quad \text{and} \quad \lim_{x \to \infty} \frac{f(x) - f(x_0)}{x - x_0} = y.
\]

According to the above remarks, it is sufficient to show that there exists a sequence of real numbers \( \{t_n\}_{n \in \mathbb{N}} \) tending to infinity, such that

\[
\lim_{n \to \infty} \chi_{t_n(E-x_0) \cap (-1,1)}(x) = \chi(-1,1)(x)
\]

I-a.e. on \((-1,1)\). Let \( t_p = (h_{n_p})^{-1} \). Then

\[
t_p \cdot (E - x_0) \supset t_p \left[ \left( -h_{n_p} - \frac{h_{n_p}}{k_{n_p}}, \frac{h_{n_p}}{k_{n_p}} \right) \cup \left( \frac{h_{n_p}}{k_{n_p}}, h_{n_p} \right) \right]
\]

\[
= \left( -1, -\frac{1}{k_{n_p}} \right) \cup \left( \frac{1}{k_{n_p}}, 1 \right).
\]

Since \( h_{n_p} \downarrow 0 \) for \( p \to \infty \) and \( k_n \not\to \infty \) for \( n \to \infty \), we have \( t_p / \to \infty \) as \( p \to \infty \) and \( 1/k_{n_1} \downarrow 0 \) as \( l \to \infty \). Hence, for \( x \in (0,1) \), \( x \neq 0 \), there exists \( p_0 \) such that, for each \( p \geq p_0 \),

\[
x \in \left( -1, -\frac{1}{k_{n_p}} \right) \cup \left( \frac{1}{k_{n_p}}, 1 \right).
\]

Then

\[
x \in t_p \cdot \left[ \left( -h_{n_p} - \frac{h_{n_p}}{k_{n_p}}, \frac{h_{n_p}}{k_{n_p}} \right) \cup \left( \frac{h_{n_p}}{k_{n_p}}, h_{n_p} \right) \right] \subset t_p \cdot (E - x_0),
\]

so, for \( p \geq p_0 \), we have

\[
\chi_{t_p(E-x_0) \cap (-1,1)}(x) = 1.
\]
Therefore \( \tilde{d}_f(E, x_0) = 1 \).

We only need to show that

\[
\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = y.
\]

Let \( \varepsilon > 0 \). There exists \( p_0 \) such that \( 1/k_{n_p} < \varepsilon/2 \) for \( p \geq p_0 \). Then, for each \( x \in E \) such that \( |x - x_0| < h_{n_{p_0}} \), we have

\[
x \in \bigcup_{p = p_0}^{\infty} \left[ \left( x_0 - h_{n_p}, x_0 - \frac{h_{n_p}}{k_{n_p}} \right) \cup \left( x_0 + \frac{h_{n_p}}{k_{n_p}}, x_0 + h_{n_p} \right) \right].
\]

From (3) it follows that there exists \( p \geq p_0 \) such that

\[
\left| \frac{f(x) - f(x_0) - A_{n_p}}{x - x_0} \right| < z_{n_p} = \frac{1}{k_{n_p}} < \frac{\varepsilon}{2}.
\]

Since \( A_{n_p} \to y \) as \( p \to \infty \), there exists \( p_1 \) such that, for \( p \geq p_1 \), we get

\[
|A_{n_p} - y| < \frac{\varepsilon}{2}.
\]

Let \( p_2 = \max(p_0, p_1) \) and \( \delta = h_{n_{p_2}} \). According to the above remarks, for \( x \in E \), if \( |x - x_0| < \delta \), then we have

\[
\left| \frac{f(x) - f(x_0) - y}{x - x_0} \right| < \varepsilon.
\]

This completes the proof.

REFERENCES


Zajićek ([3]) udowodnił, że typowa funkcja rzeczywista ma w każdym punkcie $x \in (0,1)$ istotną liczbę pochodną. W pracy tej dowodzimy, że twierdzenie to pozostaje prawdziwe, jeśli zastąpimy pojęcie istotnej liczby pochodnej przez analogiczne pojęcie dla kategorii Baire'a.

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