In this paper we study the problem of existence and uniqueness of holomorphic solution of the equation \( Df(z)(h(z)) = F(z, f(z)) \) for \( z \in B^n \) with the condition \( f(0) = 0 \) and the assumption that 0 is a singular point (i.e. \( h(0) = 0 \)).

Let \( \mathbb{C}^n \) denote the space of \( n \) complex variables \( z = (z_1, \ldots, z_n) \) with Euclidean inner product \( <z, w> = \sum_{j=1}^{n} z_j w_j \) and norm \( \|z\| = \sqrt{<z, z>} \). The ball \( \{z; \|z\| < r\} \) is denoted by \( B^n_r \). The class of holomorphic mappings from an open set \( \Omega \) into \( \mathbb{C}^n \) is denoted by \( \mathcal{H}(\Omega, \mathbb{C}^n) \). The letter \( \mathcal{I} \) represent the identity map on \( \mathbb{C}^n \). Let \( h \in \mathcal{H}(B^n_r, \mathbb{C}^n), F \in \mathcal{H}(B^n_r \times B^n_r, \mathbb{C}^n), h(0) = 0 \) and \( F(0,0) = 0 \).

The considerations concerning existence and uniqueness mapping \( f \in \mathcal{H}(B^n_{r_0}, \mathbb{C}^n) \) satisfying nonlinear generalized differential equation of the form

\[
Df(z)(h(z)) = F(z, f(z)) \text{ for } z \in B^n_{r_0}
\]

(compare [6], [7]) are presented bellow. Let

\[
\mathcal{M}_r = \{h \in \mathcal{H}(B^n_r, \mathbb{C}^n); h(0) = 0, Dh(0) = \mathcal{I}, r \epsilon <h(z), z>> 0 \text{ for } z \in B^n_r \setminus \{0\}\}.
\]
Theorem 1. Let \( h \in \mathcal{H}(B^p, \mathbb{C}^n), F \in \mathcal{H}(B^p \times B^p, \mathbb{C}^n) \), \( h(0) = 0, Dh(0) = \tilde{\xi}, F(0, y) = 0 \) for \( y \in B^p \). Let \( r_1, r_2, C, L \) be positive constants such that

(i) \( 0 < r_1 < r, \ 0 < r_2 < \rho \),

(ii) \( \|F(z, y)\| \leq C \) for \( (z, y) \in B^{n_1} \times B^{n_2} \),

\[ \|F(z, y_1) - F(z, y_2)\| \leq L\|y_1 - y_2\| \]

(iii) \( \text{for } z \in B^{n_1}, y_1, y_2 \in B^{n_2} \),

(iv) \( h \in \mathcal{M}_{r_1} \).

Then for any \( r_0 \) such that

\[ 0 < r_0 < \min\left(r_1 + \frac{C}{2r_2} - \sqrt{\frac{r_1C}{r_2} + \frac{C^2}{4r_2^2}}, r_1 + \frac{L}{2} - \sqrt{r_1L + \frac{L^2}{4}}\right) \]

the differential equation

(1) \( Df(z)(h(z)) = F(z, f(z)) \) for \( z \in B^{n_0} \)

with the condition \( f(0) = 0 \) has exactly one solution \( f \in \mathcal{H}(B^{n_0}, B^p) \).

Proof. We first observe that by Theorem 2.1 from [5] and by \( h \in \mathcal{M}_{r_1} \), the differential equation

(2) \( \frac{\partial v}{\partial t}(z, t) = -h(v(z, t)) \)

has exactly one solution \( v = v(z, t) \) defined for \( (z, t) \in B^{n_1} \times [0, \infty) \).

From theorems concerning dependence of solution of differential equation upon initial conditions (compare e.g. [1]) it follows that \( v \) is continuous on \( B^{n_1} \times [0, \infty) \) and, for any \( t \in [0, \infty), v(\cdot, t) \in \mathcal{H}(B^{n_1}, \mathbb{C}^n) \).

Next, let \( \mathcal{H}_0^\infty \) denote the space of all holomorphic bounded mappings \( f \) from \( B^p \) into \( \mathbb{C}^n \), such that \( f(0) = 0 \), with the sup norm and a closed ball, in \( \mathcal{H}_0^\infty \), with radius \( r \) and centre 0 will be denoted by \( K_r \).

Consider the mapping \( T \) defined on \( K_r \) in the following way

\[ T(f)(z) = \int_0^\infty F(v(z, t), f(v(z, t)))dt \text{ for } z \in B^{n_0} \]
where \( f \in K_{r_2} \) and \( v \) is the solution of (2).

We next prove that such definition \( T \) is correct and \( T(K_{r_2}) \subseteq K_{r_2} \). Let us first observe that for any \( y \in B^n_{r_2} \) the mapping \( F(\cdot, y) \) satisfies the assumptions of the Schwarz Lemma (see [4], Theorem 7.19, p.56).

Hence

\[
\|F(z, y)\| \leq C \frac{\|z\|}{r_1} \quad \text{for} \quad (z, y) \in B^n_{r_1} \times B^n_{r_2}.
\]

By Lemma 2.2 from [5] we get immediately

\[
\|v(z, t)\| \leq \frac{r_1 r_0 e^{-t}}{(r_1 - r_0)^2} \quad \text{for} \quad (z, t) \in B^n_{r_0} \times [0, \infty).
\]

Consequently, from (3) and (4) and by definition of \( r_0 \) we obtain

\[
\|F(v(z, t), f(v(z, t)))\| \leq C r_0 \frac{\|v(z, t)\|}{(r_1 - r_0)^2} e^{-t}
\]

\[
\quad \text{for} \quad (z, t) \in B^n_{r_0} \times [0, \infty).
\]

By the above, it follows that the definition of \( T \) is correct and \( T(K_{r_2}) \subseteq K_{r_2} \). We now show that the mapping \( T \) is contractive. Using the Schwarz Lemma and our assumptions about \( F \) we have

\[
\|T(f_1)(z) - T(f_2)(z)\| \leq \frac{L}{r_1} \|z\| \|y_1 - y_2\|
\]

\[
\quad \text{for} \quad z \in B^n_{r_1}, y_1, y_2 \in B^n_{r_2}.
\]

Let \( f_1, f_2 \in K_{r_2} \), then from (4) and (5) it follows that

\[
\|T(f_1)(z) - T(f_2)(z)\| \leq L \int_0^\infty \frac{r_0 e^{-t}}{(r_1 - r_0)^2} \|f_1(v(z, t)) - f_2(v(z, t))\| dt
\]

\[
\quad \text{for} \quad z \in B^n_{r_1}. \text{ Hence}
\]

\[
\|T(f_1) - T(f_2)\| \leq \frac{L r_0}{(r_1 - r_0)^2} \|f_1 - f_2\| \quad \text{for} \quad f_1, f_2 \in K_{r_2}
\]
and in consequence $T$ is contractive.

The Banach contraction principle (see e.g. [2], Theorem 1.1) yields that there exists exactly one the mapping $f_0 \in K_{r_2}$ which is a fixed point of $T$. Now we show that $f_0$ is a solution of (1). By the definition of $f_0$ we have

$$f_0(z) = \int_0^\infty F(v(z,t), f_0(v(z,t))) dt \quad \text{for} \quad z \in B^n_{r_0}.$$  

Since $v(v(z,t), s) = v(z, t+s)$ for $s, t \in [0, \infty)$ and $z \in B^n_{r_0}$ we conclude that

$$f_0(v(z, s)) = \int_s^\infty F(v(z,t), f_0(v(z,t))) dt$$

for $s \in [0, \infty)$ and $z \in B^n_{r_0}$. Differentiating both sides of equality (6) with respect to the parameter $s$ we obtain for $s = 0_D f_0(z)(-h(z)) = -F(z, f_0(z))$ for $z \in B^n_{r_0}$.

Hence, the mapping $f_0 : B^n_{r_0} \to B^n_{r_2}$ is a holomorphic solution of equation (1) satisfying condition $f_0(0) = 0$. The next theorem also gives a sufficient condition for the existence and uniqueness of solution of equation (1).

**Theorem 2.** Let $h \in \mathcal{H}(B^n_\rho, \mathbb{C}^n)$, $F \in \mathcal{H}(B^n_\rho \times B^n_\rho, \mathbb{C}^n)$ be such that $h(0) = 0$, $Dh(0) = 3$, $F(0, y) = y$ and $D_1 F(0, y) = 0$ for $y \in B^n_\rho$. Let $r_1, r_2, C, L$ be positive constans such that

(a) \quad 0 < r_1 < r, \quad 0 < r_2 < \rho,

(b) \quad \| F(z, y) - y \| \leq C \quad \text{for} \quad (z, y) \in B^n_{r_1} \times B^n_{r_2},

(c) \quad \| F(z, y_1) - F(z, y_2) - y_1 + y_2 \| \leq L \| y_1 - y_2 \|

\quad \text{for} \quad z \in B^n_{r_1}, y_1, y_2 \in B^n_{r_2}.$$

Then for any $r_0$ such that $0 < r_0 < \min(r_1, \alpha, \beta, \gamma)$ where

$$\alpha = \frac{1}{2} \left( 2r_1 + \sqrt{L} - \sqrt{4r_1 \sqrt{L} + L} \right).$$
\[
\beta = \frac{1}{2} \left( 2r_1 + \frac{r_1}{r_2} - \sqrt{\frac{4r_1^2}{r_2} + \left(\frac{r_1}{r_2}\right)^2} \right)
\]
\[
\gamma = \frac{1}{2} \left( 2r_1 + \frac{1}{t_0} - \sqrt{(2r_1 + \frac{1}{t_0})^2 - 4r_1^2} \right),
\]
\[
t_0 = -r_1 + \frac{\sqrt{r_1^2 + 4Cr_2}}{2C},
\]

the differential equation

\[
Df(z)(h(z)) = F(z, f(z)) \quad \text{for } z \in B^n_{r_0}
\]

with the conditions \(f(0) = 0, Df(0) = \mathbb{0}\) has exactly one solution \(f \in \mathcal{H}(B^n_{r_0}, B^n_{r_2})\).

Proof. Let, as in the proof of theorem 1, \(v = v(z, t)\), for \((z, t) \in B^n \times [0, \infty)\), be a solution of equation (2). By Theorem 2 from [3] the function \(g\) defined by equality

\[
g(z) = \lim_{t \to \infty} \left( e^t v(z, t) \right) \quad \text{for } z \in B^n_r
\]

belongs to \(\mathcal{H}(B^n_r, \mathbb{C}^n)\). Let \(\mathcal{H}_0^n\) be defined as in the proof of the previous theorem and let \(K(g, \tau)\) denote a closed ball, in \(\mathcal{H}_0^n\), with radius \(\tau\) and centre \(g\). Assume that \(0 < \tau < r_2 - \frac{r_1 r_0}{(r_1 - r_0)^2}\). Next, consider the integral operator \(T\) of the form

\[
T(f)(z) = g(z) + \int_0^\infty e^t |F(v(z, t), f(v(z, t))) - f(v(z, t))| dt
\]

for \(z \in B^n_{r_0}\) and \(f \in K(g, \tau)\). Observe that by Theorem 7.19 from [4]

(7) \[\|F(z, y) - y\| \leq C \frac{\|z\|^2}{r_1^2} \quad \text{for } (z, y) \in B^n_{r_1} \times B^n_{r_2}.\]

Consequently, from (7) and (4) we have

(8) \[\|F(v(z, t), f(v(z, t))) - f(v(z, t))\| \leq C \frac{r_1^2 e^{-2t}}{(r_1 - r_0)^4}\]
for \((z, t) \in B^n_{r_0} \times [0, \infty)\). By the definition of \(g\) and by (4) we get
\[
\|g(z)\| \leq \frac{r_1f_0}{(r_1 - r_0)^2} \quad \text{for} \quad z \in B^n_{r_0}.
\]

From the above inequality, the definition \(r_0\) and by inequality (8) it follows that the mapping \(T\) is correctly defined and maps \(K(g, \tau)\) into \(K(g, \tau)\). Now we show that \(T\) is contractive. Using the Schwarz Lemma and our assumptions about \(F\) we obtain
\[
\|F(z, y_1) - F(z, y_2) - y_1 + y_2\| \leq L\|y_1 - y_2\| \frac{\|z\|^2}{r_1^2}
\]
for \(z \in B^n_{r_0}\) and \(y_1, y_2 \in B^n_{r_2}\). From this and (4) we have
\[
\|T(f_1)(z) - T(f_2)(z)\| \leq L \int_0^\infty \frac{r_2^2e^{-t}}{(r_1 - r_0)^4} \|f_1(v(z, t)) - f_2(v(z, t))\|\,dt
\]
for \(z \in B^n_{r_0}\) and \(f_1, f_2 \in K(g, \tau)\). Since \(\frac{Lr_2^2}{(r_1 - r_0)^4} < 1\), therefore from (9) it follows immediately that the mapping \(T\) is contractive. Hence, by the Banach contraction principle, there exists exactly one \(f_0 \in K(g, \tau)\) being a fixed point of the mapping \(T\). Next, we prove that \(f_0\) is a solution of (1). By the definition of \(f_0\) we have
\[
f_0(z) = g(z) + \int_0^\infty e^t[F(v(z, t), f_0(v(z, t))) - f_0(v(z, t))]\,dt
\]
for \(z \in B^n_{r_0}\). Since \(v(v(z, t), s) = v(z, t + s)\) for \(s, t \in [0, \infty)\) and \(z \in B^n_{r_0}\) therefore from (10) it follows that
\[
f_0(v(z, s)) = g(v(z, s)) + \int_s^\infty e^{-t}[F(v(z, t), f_0(v(z, t))) - f_0(v(z, t))]\,dt
\]
for \(s \in [0, \infty)\) and \(z \in B^n_{r_0}\). Differentiating both sides of this equality with respect to the parameter \(s\) we obtain for \(s = 0\)
\[
Df_0(z)(-h(z)) = Dg(z)(-h(z))
\]
\[
- \int_0^\infty e^t[F(v(z, t), f_0(v(z, t))) - f_0(v(z, t))]\,dt
\]
\[-F(z, f_0(z)) + f_0(z)
\]
for \(z \in B^n_{r_0}\). As \(Dg(z)(h(z)) = g(z)\) for \(z \in B^n_{r_0}\) (compare [3], Theorem 4), the above equality and (10) gives that \(f_0\) is a solution of (1).
REFERENCES


Tadeusz Poreda

O HOLOMORFICZNYCH ROZWIĄZANIACH UOGÓLNIONYCH RÓWNAŃ RÓŻNICZKOWYCH

W tej pracy badane jest istnienie i jednoznaczność holomorficznego rozwiązania równania \( Df(z)(h(z)) = F(z, f(z)) \) dla \( z \in B^n \) przy warunku \( f(0) = 0 \) i przy założeniu, że 0 jest punktem osobliwym (tzn. \( h(0) = 0 \)).

Institute of Mathematics
Lódź Technical University
al. Politechniki 11, 1-2
90-924 Lódź, Poland