Let $X$ be an infinite set. For ideals $I, J \subseteq P(X)$ and a family $F \subseteq P(X)$, we give conditions guaranteeing the existence of an $f : X \rightarrow X$ which is constant on $X \setminus C$ for some $C \in J$ and fulfills the condition: (*) $f^{-1}([x]) \cap V \notin I$ for any $x \in X$ and $V \in F$.

The result and its proof are related to the investigations made by H.I. Miller and W. Poreda. In the case when $X$ forms a perfect Polish space and $F$ consists of all nonvoid open sets, we study ideals $I$ admitting an $f : X \rightarrow X$ which satisfies (*) and is Borel measurable.

1. Introduction

Carathéodory showed in 5 that there exists a Lebesgue measurable function $f : \mathbb{R} \rightarrow \mathbb{R}$ such that $f^{-1}[E] \cap U$ has positive measure for each set $E$ of positive measure and each nondegenerate interval $U$. A modified version employing the Baire category was obtained by H. Miller in [7]. He proved the existence of a Lebesgue measurable $f : \mathbb{R} \rightarrow \mathbb{R}$ such that $f^{-1}[E] \cap U$ is of the second category for each set $E$ of second category and each nondegenerate interval $U$. He even obtained (in ZFC) a stronger result where $E$ in $f^{-1}[E] \cap U$ is replaced by $\{x\}$ (for any $x \in \mathbb{R}$). The same was shown in [11] in a different way (Continuum Hypothesis used there can be removed...
which was observed by K.P.S. Bhashara Rao in [3]). In Section 2 we prove a more general result with the help of a mixed method joining the tricks from [7] and [11]. In particular, we get a simple proof in ZFC, good for the measure and category cases. Since there is no uncountable disjoint family of measurable sets of positive measure (this is the so-called countable chain condition, abbr. ccc), there is no Lebesgue measurable \( f : \mathbb{R} \to \mathbb{R} \) such that \( f^{-1}(\{x\}) \) has positive measure for each \( x \in \mathbb{R} \). The analogous observation can be done for the category case. However, there are natural examples of ideals \( J \) (which do not satisfy ccc) admitting a Borel measurable \( f : \mathbb{R} \to \mathbb{R} \) whose all fibres are large (i.e. not in \( I \)). That property, called \( (M) \), was introduced in [2] for ideals of subsets of a perfect Polish space. In Section 2 of the present paper, we study a stronger property, called \( (M^*) \), which requires the fibres of \( f \) to be large on each nonvoid open set.

In general, we consider ideals \( I \) of subsets of an infinite set \( X \) and always assume that \( X \notin I \). A subfamily \( H \) of \( I \) is called a base of \( I \) if each \( A \in I \) is contained in some \( B \in H \). We say that two ideals \( I \) and \( J \) are orthogonal if there are \( B \in I \) and \( C \in J \) such that \( B \cup C = X \).

## 2. Remarks on Miller's Result

Recall the following theorem due to Abian and Miller (see [1] and [7]) which generalizes the result of [12].

**Theorem 2.1.** Let \( X \) be a set of infinite cardinality \( \kappa \). Let \( A \) be a family of at most \( \kappa \) subsets of \( X \), each having cardinality \( \kappa \). Denote by \( \Delta(A) \) the family of all \( D \subseteq X \) such that \( U \cap D \neq \emptyset \) for each \( U \in A \). Then, for each cardinal \( \lambda \leq \kappa \), the set \( X \) can be expressed as the union of \( \lambda \) pairwise disjoint sets belonging to \( \Delta(A) \).

**Theorem 2.2.** Assume that \( I \) and \( J \) are orthogonal ideals of subsets of a set \( X \) of cardinality \( \kappa \). Let \( I \) have a base \( H \) of size \( \leq \kappa \) and let \( F \subseteq P(X) \) be a given family of size \( \leq \kappa \) such that \( |V \setminus E| = \kappa \) for any \( V \in F \) and \( E \in H \). Then, for each \( x_0 \in X \), there are a set \( C \in J \) and a function \( f : X \to X \) such that \( f(x) = x_0 \) for each \( x \in X \setminus C \).
and

\[ f^{-1}([x]) \cap V \notin I \text{ for any } x \in X \text{ and } V \in F. \]

**Proof.** Put \( A = \{ V \setminus E : V \in F \text{ and } E \in H \} \) and apply Theorem 1.1 to it. Then \( X \) can be expressed as the union of a disjoint family \( \Delta^* \subseteq \Delta(A) \) of size \( \kappa \). Let \( X = B \cup C \) where \( B \in I, C \in J \) and \( B \cap C = \emptyset \). Choose any bijection \( h : \Delta^* \to X \) and define \( f : X \to X \) as follows. If \( x \in B \), put \( f(x) = x_0 \), and if \( x \notin B \), choose a unique \( D_x \in \Delta^* \) such that \( x \in D_x \) and put \( f(x) = h(D_x) \). Then, obviously, \( f(x) = x_0 \) for \( x \in X \setminus C \). If \( x \in X \), then

\[
f^{-1}([x]) = \begin{cases} h^{-1}(x) \setminus B & \text{for } x \neq x_0, \\ h^{-1}(x) \cup B & \text{for } x = x_0. \end{cases}
\]

Consider any \( V \in F \). Observe that \( V \cap D \notin I \) for each \( D \in \Delta(A) \). Indeed, if \( V \cap D \in I \) for some \( D \in \Delta(A) \), we choose \( E \in H \) such that \( V \cap D \subseteq E \). We infer that \( V \setminus E \in A \) and \( (V \setminus E) \cap D = \emptyset \), which contradicts the assumption \( D \in \Delta(A) \). Now, taking \( D = h^{-1}(x) \), we have \( h^{-1}(x) \cap V \notin I \). Since \( B \in I \), we get \( f^{-1}([x]) \cap V \notin I \).

In particular, let \( X = \mathbb{R} \) and let \( I \) (resp. \( J \)) be the ideal of all Lebesgue null sets (resp. meager sets) in \( \mathbb{R} \). It is well known that the family \( H \) of all \( G_\delta \) null sets (resp. \( F_\sigma \) meager sets) forms a base of \( I \) (resp. \( J \)), its cardinality equals \( c = |\mathbb{R}| \), and \( |V \setminus E| = c \) for any open \( V \neq \emptyset \) and \( E \in H \). Moreover, \( I \) and \( J \) are orthogonal (see [10]).

Thus from Theorem 2.2 we derive

**Corollary 2.3.** (a) There is an \( f : \mathbb{R} \to \mathbb{R} \) such that \( \{ x \in \mathbb{R} : f(x) \neq 0 \} \) is meager (thus \( f \) has the Baire property) and \( f^{-1}([x]) \cap V \) has positive outer measure for any \( x \in \mathbb{R} \) and open \( V \neq \emptyset \).

(b) (see [7], [11]). There is an \( f : \mathbb{R} \to \mathbb{R} \) such that \( \{ x \in \mathbb{R} : f(x) \neq 0 \} \) is a null set (thus \( f \) is Lebesgue measurable) and \( f^{-1}([x]) \cap V \) is of the second category for any \( x \in \mathbb{R} \) and open \( V \neq \emptyset \).

Another interesting pair of orthogonal ideals to which Theorem 2.2 can be applied is described in [9], Proposition 5.
3. Property \((M^*)\)

Now, we add the requirement of the Borel measurability of \(f\) to condition \((*)\) formulated in Theorem 2.2. Let \(X\) be a perfect Polish space and \(I\) - an ideal of subsets of \(X\). We say (cf. [2]) that \(I\) has property \((M)\) (resp. property \((M^*)\)) if there is a Borel measurable function \(f : X \to X\) such that \(f^{-1}([x]) \notin I\) for each \(x \in X\) (resp. \(f^{-1}([x]) \cap V \notin I\) for any \(x \in X\) and open \(V \neq \emptyset\)). We then say that \(f\) realizes \((M)\) (resp. \((M^*)\)) for \(I\). Obviously, \((M^*)\) implies \((M)\). We shall show that the converse is false (Example 3.5).

Remarks. (a) If \(I\) and \(J\) are ideals of subsets of \(X\) such that \(I \subseteq J\) and \(J\) has \((M)\) (resp. \((M^*)\)), then \(I\) has \((M)\) (resp. \((M^*)\)).

(b) Since any two perfect Polish space are Borel isomorphic (see [8], 1 G4), we may replace \(f : X \to X\) in the definition of \((M)\) and \((M^*)\) by \(f : X \to Y\) for a suitable perfect Polish \(Y\).

In [2], several examples of ideals with property \((M)\) are given. Our aim is to find nontrivial ideals with property \((M^*)\).

It was noticed in [4], Ex. 1.3, p. 4, that there exists a Borel function \(f\) from \((0,1)\) into \((0,1)\) such that \(f^{-1}([x])\) is dense for each \(x \in (0,1)\) (this was treated as a strong version of the Darboux property). The same can be inferred from [2], Th. 3.4, p. 44, where another method leads to a Borel mapping from a perfect Polish space \(X\) onto the Cantor space, with all fibres dense in \(X\). In fact, the existence of such a mapping implies that the ideal of all nowhere dense sets in \(X\) has property \((M^*)\). Our next example of an ideal with property \((M^*)\) is also derived from [2]. It turns out that the respective proof for \((M)\) given in [2] (generalizing Mauldin's construction from [6]) works for \((M^*)\), but some parts require a more detailed analysis which will be done below.

**Theorem 3.2** (cf.[2], Th.3.3, p. 42). Let \(I\) be a \(\sigma\)-ideal of subsets of a perfect Polish space \(X\). Assume that \(I\) contains all singletons, does not contain nonempty open sets and has a base consisting of \(G_\delta\) sets. Then the \(\sigma\)-ideal \(J\) of all sets that can be covered by \(F_\alpha\) sets from \(I\) has property \((M^*)\).

A nonempty closed set \(F \subseteq X\) will be called \(I\)-perfect if \(F \cap V \neq \emptyset\) implies \(F \cap V \notin I\) for any open \(V \subseteq X\).
Let us explain some notation. Let \( \omega = \{0,1,2,\ldots\} \). By \( 2^{<\omega} \) and \( 2^\omega \) we denote, respectively, the sets of all finite and infinite sequences of zeros and ones. The empty sequence (which also belongs to \( 2^{<\omega} \)) will be written as \( (\)\). By \( s_0 \) and \( s_1 \) we denote the respective extensions of \( s \in 2^{<\omega} \). For \( z \in 2^\omega \) and \( n \in \omega \), put \( z|n = (z(0), z(1), \ldots, z(n-1)) \). The set \( 2^\omega \), endowed with the product topology, is called the Cantor space. It forms a perfect Polish space.

The following lemma results immediately from the construction given in [2], pp. 42-43.

**Lemma 3.3.** Under the assumptions of Theorem 3.2, there is a family \( \{C^n_s : s \in 2^{<\omega}, n \in \omega\} \) of I-perfect sets with the properties:

1. For each nonempty open \( V \subseteq X \), there is an \( n \in \omega \) such that \( C^n_s \subseteq V \);
2. For any \( s \in 2^{<\omega}, n \in \omega \) and a nonempty \( V \) relatively open in \( C^n_s \), there is an \( m \in \omega \) such that \( C^m_{s0} \cup C^m_{s1} \subseteq V \);
3. For any \( s \in 2^{<\omega} \) and \( m \in \omega \), the condition \( C^m_{s0} \cap C^m_{s1} = \emptyset \) holds and there is an \( n \in \omega \) such that \( C^m_{s0} \cup C^m_{s1} \subseteq C^n_s \).

**Lemma 3.4.** Under the assumptions of Theorem 3.2, if a family \( \{C^n_s : s \in 2^{<\omega}, n \in \omega\} \) fulfills conditions (1)-(2) of Lemma 3.3, then, for any \( z \in 2^\omega \), a set \( H \in J \) and a nonempty open \( V \subseteq X \), there exists a sequence \( \{n_i : i \in \omega\} \) of nonnegative integers such that

\[
\emptyset \neq \bigcap_{i \in \omega} C^{n_i}_{z|i} \subseteq V \setminus H.
\]

**Proof.** Since \( H \in J \), there is a sequence of closed sets \( F_n \subseteq I \) such that \( H \subseteq \bigcup_{n \in \omega} F_n \). The set \( V \setminus F_0 \) is open and nonempty (in fact, \( V \setminus F_0 \notin I \) since \( V \notin I \) and \( F_0 \in I \)). By (1), pick \( n_0 \in \omega \) so that \( C^{n_0}_s \subseteq V \setminus F_0 \). For any \( i \in \omega \), having \( n_i \) chosen, pick \( n_{i+1} \in \omega \) so that \( C^{n_{i+1}}_{z|i+1} \subseteq C^{n_i}_{z|i} \setminus F_{i+1} \) (we use (2)); here \( C^{n_i}_{z|i} \setminus F_{i+1} \) is nonempty (in fact, it does not belong to \( I \)) and relatively open in \( C^{n_i}_{z|i} \). From the classical Cantor theorem we get \( C = \bigcap_{i \in \omega} C^{n_i}_{z|i} \neq \emptyset \). Of course, \( C \) is disjoint from \( \bigcup_{n=1}^{\infty} F_n \) and, consequently, from \( H \).
Proof of Theorem 3.2. We use the sets $C^a_s$ from Lemma 2.3. Put $C_s = \bigcup_{n \in \omega} C^a_s$ for $s \in 2^{<\omega}$. Then we have

(a) $C_{s_0} \cap C_{s_1} = \emptyset$ for all $s \in 2^{<\omega}$,
(b) $C_{s_0} \cup C_{s_1} \subseteq C_s$ for all $s \in 2^{<\omega}$,

which follows from (3). Define $B = \bigcap_{n \in \omega} \bigcup_{z \in 2^n} C_{z|n}$. It is not hard to prove (see [2]) that:

(c) $B$ is a Borel set;
(d) for each $x \in B$, there is a unique $h(x) \in 2^\omega$ such that $x \in \bigcap_{n \in \omega} C_{h(x)|n}$;
(e) the function $h : B \to 2^\omega$ defined in (d) is Borel measurable;
(f) $h^{-1}([z]) = \bigcap_{n \in \omega} C_{z|n}$ for each $z \in 2^\omega$.

Let $g : X \to 2^\omega$ be a fixed Borel measurable extension of $h$. By (f), we have $g^{-1}([z]) \supseteq \bigcap_{n \in \omega} C_{z|n}$. Consider any nonempty open $V \subseteq X$. It suffices to show that $V \cap \bigcap_{n \in \omega} C_{z|n} \notin J$. Suppose that $V \cap \bigcap_{n \in \omega} C_{z|n} = H \in J$. According to Lemma 3.4, there is a sequence $(\langle r_i : i \in \omega \rangle)$ for which $\emptyset \neq \bigcap_{i \in \omega} C_{z_{\langle r_i \rangle}} \subseteq V \setminus H$. On the other hand

$$V \cap \bigcap_{n \in \omega} C^r_{z_{\langle r_i \rangle}} \subseteq V \cap \bigcap_{i \in \omega} C_{z|n} = H,$$

a contradiction.

By Theorem 3.2, the ideal of sets that can be covered by $F_\sigma$ Lebesgue null sets has $(M^*)$.

Example 3.5. Let $I$ consist of all sets $A \subseteq \mathbb{R}$ such that $A \cap (-\infty, 0)$ is of Lebesgue measure zero and $A \cap [0, \infty)$ is contained in an $F_\sigma$ set of measure zero. Then $I$ forms a $\sigma$-ideal of subsets of $\mathbb{R}$. Observe that $I$ has property $(M)$. Indeed, the family $I_+ = \{A \in I : A \subseteq [0, \infty)\}$ is a $\sigma$-ideal of subsets of $X = [0, \infty)$, which fulfils the assumptions of Theorem 3.2. Hence it has property $(M^*)$ and, consequently, property $(M)$ (in $X$). Let $f_+: X \to \mathbb{R}$ realize property $(M)$ for $I_+$. If we extend $f_+$ to a Borel $f : \mathbb{R} \to \mathbb{R}$, then $f$ realizes property $(M)$ for $I$. On the other hand, $I$ has not $(M^*)$. Indeed, suppose that $g : \mathbb{R} \to \mathbb{R}$ realizes $(M^*)$ for $I$. Then $\{g^{-1}([y]) \cap (-\infty, 0) : y \in \mathbb{R}\}$ forms an uncountable disjoint family of Borel sets with positive measure, which is impossible.
In the above example, $I$ is not translation-invariant, i.e. the condition

$$A + x \in I \text{ for any } A \in I \text{ and } x \in \mathbb{R},$$

where $A + x = \{ a + x : a \in A \}$, is not fulfilled. So, it would be interesting to find an example omitting that fault.

Let us note that Example 3.5 essentially uses the fact that property $(M)$ (unlike $(M^*)$) need not be hereditary with respect to open sets. To be more precise, let us say that an ideal $I$ has property $(M')$ if $I \cap P(V)$ has $(M)$ (in $V$) for any nonvoid open $V \subseteq X$. Obviously, $(M^*) \Rightarrow (M') \Rightarrow (M)$. Our example shows, in fact, that $(M) \Rightarrow (M')$ is false. This suggests the question whether $(M') \Rightarrow (M^*)$ must hold.

**Acknowledgement.** I would like to thank W. Poreda for her interest and valuable remarks.

**References**


Niech $X$ będzie zbiorem nieskończonym. Dla pewnych idealów $I, J \subseteq P(X)$ i rodziny $F \subseteq P(X)$ uzyskano warunki dostateczne istnienia funkcji $f : X \rightarrow X$ stałą na $X \setminus C$ dla pewnego $C \in J$ oraz spełniającej warunek: $(*) \ f^{-1}([x]) \cap V \notin I$ dla dowolnych $x \in X$ i $V \in F$. Wynik i jego dowód wiąże się z wcześniejszymi badaniami H. Millera i W. Poredy. W przypadku gdy $X$ jest doskonałą przestrzenią polską oraz $F$ składa się z niepustych zbiorów otwartych, badamy ideały $I$, dla których istnieje borelowska funkcja $f : X \rightarrow X$ spełniająca $(*)$.