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GENERALIZATIONS OF TUKEY-LAMBDA DISTRIBUTIONS

Abstract. The Generalized Lambda Distribution (GLD) is a four-parameter generalization of Tukey’s Lambda family. Several methods for estimating the parameters of the GLD have been reported in the literature, but the most popular is the moment-method matching proposed by Ramberg and Schmeiser (1974). One criticism of the GLD referred to above is that the shape parameters also determine skewness. It seems reasonable that there should be three linear parameters determining position, scale, and skewness and two parameters determining the shapes of the two tails. This suggests a natural generalization of the GLD to give a five-parameter lambda distribution (FPLD). The aim of paper is to show that the GLD and the FPLD describe empirical distribution quite well.

Key words: generalized lambda distribution, estimation methods, distribution fitting, quotations.

I. THE GENERALIZED \( \lambda \) DISTRIBUTION

One of the more important tasks in statistical data analysis is fitting a probability distribution. We usually search for distribution parameters through fitting a probability density function (PDF) or a cumulative distribution function (CDF) to some data. In the case of Tukey-lambda distribution we search for distribution parameters on the basis of the percentile function which is the inverse of the cumulative distribution function.

An example of the percentile function is Tukey’s \( \lambda \) function

\[
R(p) = \frac{p^\lambda - (1-p)^\lambda}{\lambda}, \quad 0 \leq p \leq 1, \quad \lambda \neq 0.
\]  

(1)

This function depends on one parameter, so it describes only a family of symmetrical continuous probability distributions. For example,

- \( \lambda = -1 \) approximately Cauchy
- \( \lambda = 0 \) exactly logistic

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• \( \lambda = 0.14 \) approximately normal
• \( \lambda = 0.5 \) U-shaped
• \( \lambda = 1 \) exactly uniform \((x \in (-1,1))\).

Ramberg and Schmeiser (1974) generalized Tukey’s \( \lambda \) distribution to a four-parameter distribution defined by the percentile function

\[
R(p) = \lambda_1 + \frac{p^{\lambda_3} - (1 - p)^{\lambda_4}}{\lambda_2}, \quad 0 \leq p \leq 1,
\]

where \( \lambda_1 \) is a location parameter, \( \lambda_2 \) is a scale parameter, \( \lambda_3 \) is a skewness parameter and \( \lambda_4 \) is a kurtosis one.

The percentile function is the inverse of the cumulative distribution function, so if \( X \) is a continuous random variable with percentile function \( R \) and \( p \) is a uniform random variable on the interval \([0,1]\), then \( X = R(p) \) and CDF of variable \( X \) has form

\[
F(R(p)) = p.
\]

Differentiating both sides of the equation (3) with respect to \( p \) yields the expression for density function

\[
f(x) = \frac{1}{dR(p)} = \frac{\lambda_2}{\lambda_3 p^{\lambda_3 - 1} + \lambda_4 (1 - p)^{\lambda_4 - 1}}, \quad 0 \leq p \leq 1,
\]

which depends only on three parameters. Unfortunately, the CDF does not exist in analytic form.

The generalized \( \lambda \) distribution defines the family of asymmetrical probability distributions of continuous variable, inter alia: Gamma distribution, Beta or Weibull distribution. Besides, the most important thing is that, if we search for parameters of the distribution, we do not have to know which theoretical distribution will better describe the empirical data.

The density function of GlD is not proper function for all combinations of values of shape parameters. Ramberg and Schmeiser (1974) indicated four regions of parameters which produce a proper statistical distribution (Table 1); however Karian, Dudewicz and McDonald (1996) characterized two new regions. Figure 1 shows all allowed regions of \( \lambda_3 \) and \( \lambda_4 \) parameter space.
Generalizations of Tukey-Lambda Distributions

Table 1. Range of \( \lambda \) values for Ramberg’s parametrization of the G\( \lambda \)D

<table>
<thead>
<tr>
<th>Region</th>
<th>( \lambda_1 )</th>
<th>( \lambda_2 )</th>
<th>( \lambda_3 )</th>
<th>( \lambda_4 )</th>
<th>Minimum</th>
<th>Maximum</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>all</td>
<td>&lt; 0</td>
<td>&lt; -1</td>
<td>&gt; 1</td>
<td>( -\infty )</td>
<td>( \lambda_1 + (1/\lambda_2) )</td>
</tr>
<tr>
<td>2</td>
<td>all</td>
<td>&lt; 0</td>
<td>&gt; 1</td>
<td>&lt; -1</td>
<td>( -\infty )</td>
<td>( \lambda_1 + (1/\lambda_2) )</td>
</tr>
<tr>
<td>3 all</td>
<td>&gt; 0</td>
<td>0</td>
<td>&gt; 0</td>
<td>( -\infty )</td>
<td>( \lambda_1 - (1/\lambda_2) )</td>
<td>( \lambda_1 + (1/\lambda_2) )</td>
</tr>
<tr>
<td>4 all</td>
<td>&lt; 0</td>
<td>&lt; 0</td>
<td>&lt; 0</td>
<td>( -\infty )</td>
<td>( \lambda_4 )</td>
<td>( \lambda_4 )</td>
</tr>
</tbody>
</table>


Fig. 1. Regions of the \( \lambda_3 \) and \( \lambda_4 \) plane that produce proper statistical distributions in the G\( \lambda \)D

II. METHODS FOR ESTIMATING THE PARAMETERS OF THE G\( \lambda \)D

Several methods for estimating the parameters of the G\( \lambda \)D have been reported in the literature (Lakhany and Mausser (2000), Taristano (2010), Taristano (2004), King and MacGillivray (1999), Karvanen and Nuutinen (2007), Su (2007)); we describe briefly some of them.
The moment-matching method was proposed by Ramberg and Schmeiser (1979). This method can be described as follows. First, we find the expressions for the mean, the variance, and the third and fourth moments of $G\lambda D$:

\[ \mu = \lambda_1 + \frac{1}{\lambda_2} \left( \frac{1}{\lambda_3 + 1} - \frac{1}{\lambda_4 + 1} \right) \]

\[ \sigma^2 = \frac{1}{\lambda_2^2} (A_2 - A_1^2) \]

\[ \beta_3 = \frac{1}{(A_2 - A_1^2)^3} \left( A_3 - 3 A_1 A_2 + 2 A_1^3 \right) \]

\[ \beta_4 = \frac{1}{(A_2 - A_1^2)^4} \left( A_4 - 4 A_1 A_3 + 6 A_1^2 A_2 + 3 A_1^4 \right) \]

where

\[ A_1 = \frac{1}{\lambda_3 + 1} - \frac{1}{\lambda_4 + 1} \]

\[ A_2 = \frac{1}{2\lambda_3 + 1} + \frac{1}{2\lambda_4 + 1} - 2\beta(\lambda_3 + 1, \lambda_4 + 1) \]

\[ A_3 = \frac{1}{3\lambda_3 + 1} - \frac{1}{3\lambda_4 + 1} - 3\beta(2\lambda_3 + 1, \lambda_4 + 1) + 3\beta(\lambda_3 + 1, 2\lambda_4 + 1) \]

\[ A_4 = \frac{1}{4\lambda_3 + 1} + \frac{1}{4\lambda_4 + 1} - 4\beta(3\lambda_3 + 1, \lambda_4 + 1) + 6\beta(2\lambda_3 + 1, 2\lambda_4 + 1) - 4\beta(\lambda_3 + 1, 3\lambda_4 + 1) \]

Next we equate the mean, the variance, and the third and fourth moments of $G\lambda D$ to the corresponding mean $\mu^*$, variance $(\sigma^*)^2$, skewness $\beta_3^*$, and kurtosis $\beta_4^*$ of the sample. Finally, we compute the parameters $\lambda$ from equations

\[ \mu = 0 \]

\[ \sigma^2 = 1 \]

\[ \beta_3 = \beta_3^* \]

\[ \beta_4 = \beta_4^* \]
Ramberg at al (1979) presented the tables of the parameters $\lambda$ for selected values of $\beta^*_a$, $\beta^*_b$ and $\mu = 0$, $\sigma^2 = 1$.

2) The least squares method was proposed by Öztürk and Dale (1985). This method can be described as follows. Let $x_i$, $i = 1, ..., n$ denote the $i$th order statistic of data which is to be represented by the quantile function $R(u)$ and let $U_i$, $i = 1, ..., n$ denote the order statistic of the corresponding uniformly distributed random variable. The method finds values of $\lambda$ for which the differences between the observed and predicted order statistics are as small as possible. So, we must minimize the function

$$G(\lambda) = \sum_{i=1}^{n} \left( x_i - \lambda - \frac{Z_i}{\lambda} \right)^2$$  \hspace{1cm} (7)

where:

$$Z_i = \frac{1}{\lambda + \lambda_a} \left( EU_{(i)}^{\lambda_a} - 1 \right) - \frac{1}{\lambda} \left( E(1-U_{(i)})^{\lambda} - 1 \right)$$

$$EU_{(i)}^{\lambda_a} = \frac{\Gamma(n+1)\Gamma(i+\lambda_a)}{\Gamma(i)\Gamma(n+\lambda_a+1)}$$

$$E(1-U_{(i)})^{\lambda} = \frac{\Gamma(n+1)\Gamma(n-i+\lambda_a+1)}{\Gamma(n-i+1)\Gamma(n+\lambda_a+1)}$$

3) The starship method can be described in a few steps:

   Step 1. We select a region in four-dimensional space that covers the range of the four parameters $\lambda$.

   Step 2. On the region selected in Step 1, we overlay a four-dimensional rectangular grid.

   Step 3. We evaluate the grid points created in Step 3 by performing a goodness-of-fit test on the corresponding distributions. If the test is satisfied, we stop procedure; otherwise we continue with the next point in the grid.

   Many articles concerning the G$\lambda$D are devoted to the comparison of methods of estimating parameters. Authors show that the moment matching method and the least square method fit probability distribution into empirical data better than the starship method. For that reason, we use only the first two estimate methods in our calculations.
III. APPLICATIONS

In this section we focus our attention on the application of GλD to stock market quotations. The first data concern daily quotations of WIG20 index from the time period 03.01.1994-26.10.2012. Using the moment matching and the least square methods we calculate the parameters of GλD. The first four moments of the data are:

\[
\begin{align*}
\mu^* &= 1901.203 \\
\beta_3^* &= 0.6 \\
\sigma^* &= 733.404 \\
\beta_4^* &= 2.74
\end{align*}
\]

In moment matching method for the values of skewness and kurtosis, mentioned above, we are inside the range of tabulated values in paper Ramberg at all (1979), so the lambda values were read directly from the table (Ramberg at all (1979)). However, in the least square method we found the lambda values from equation (7). Table 2 includes the results of both methods.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Estimate method</th>
<th>Moment matching</th>
<th>Least squares</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\lambda_1)</td>
<td></td>
<td>1188.34</td>
<td>1177.24</td>
</tr>
<tr>
<td>(\lambda_2)</td>
<td></td>
<td>0.0003</td>
<td>0.0003</td>
</tr>
<tr>
<td>(\lambda_3)</td>
<td></td>
<td>0.0355</td>
<td>0.0346</td>
</tr>
<tr>
<td>(\lambda_4)</td>
<td></td>
<td>0.3265</td>
<td>0.3357</td>
</tr>
</tbody>
</table>

Source: own calculations.

Fig. 2. The histogram of daily quotations of WIG20 index and corresponding GλD (red curve – the solution for the moment matching method, black curve – the solution for the least squares method)
Figure 2 illustrates the results of fitting a G\(\lambda\).D to the quotation distribution of WIG20 index. The difference between the two G\(\lambda\).Ds is almost imperceptible. Ramberg at all (1979) calculated \(\chi^2\) goodness-of-fit statistics to check that the model fits the data well. In our cases the computed value of \(\chi^2\) is large, so we must reject the solutions for the quotation of WIG20 index. Of course, we should expect these results. Besides, these data form a very large set, so in the next example we chose a smaller set of data than the first.

Now we consider daily quotations of Nikkei 225 index from the time period 18.08.2003-29.07.2005. The first four moments of the data read:

\[
\begin{align*}
\mu^* &= 11135.54 & \beta_3^* &= -0.26 \\
\sigma^* &= 486.97 & \beta_4^* &= 2.47
\end{align*}
\] (9)

We found the distribution parameters of quotations of Nikkei 225 index in the same way as the WIG20 index. Table 3 shows the results of fitting G\(\lambda\).D.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Estimate method</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\lambda_1)</td>
<td>Moment matching</td>
</tr>
<tr>
<td>(\lambda_2)</td>
<td>10750.8</td>
</tr>
<tr>
<td>(\lambda_3)</td>
<td>0.00058</td>
</tr>
<tr>
<td>(\lambda_4)</td>
<td>0.0843</td>
</tr>
<tr>
<td>(\lambda_5)</td>
<td>0.4294</td>
</tr>
</tbody>
</table>

Source: own calculations.

In this case the goodness of fit test rejected both solutions but the computed values of \(\chi^2\) were smaller than WIG20 case. This result may suggest that G\(\lambda\).D does not describe well the asymmetrical distributions. It could be caused by the fact that the shape parameter also determine skewness, so maybe there should be three linear parameters determining position, scale, and skewness and two parameters determining the shapes of the two tails. This suggests a natural generalization of the G\(\lambda\).D to give a five-parameter lambda distribution (FPLD).

The five-parameter version of generalized lambda distribution was proposed by Gilchrist (2000). The quantile function is given by

\[
Q(x) = \lambda_1 + \frac{\lambda_2}{2} \left( \frac{1 - \lambda_3}{\lambda_4} P^\lambda_1 - 1 \right) - \frac{1 + \lambda_3}{\lambda_5} \left( 1 - P \right) \left( 1 - P \right) - 1
\] (9)
where \( 0 \leq p \leq 1, \ \lambda_2 \geq 0, \ \lambda_3 \in \{-1, 1\} \). According to the method proposed by Öztürk and Dale (1985), we must minimize the function

\[
G(\lambda) = \sum_{i=1}^{n} (x(i) - Z(i))^2
\]

(10)

where

\[
Z(i) = \lambda_1 + \frac{(1 - \lambda_3) \lambda_2}{2 \lambda_4} \left( \frac{\Gamma(n+1)\Gamma(i+\lambda_4)}{\Gamma(i)\Gamma(n+1+\lambda_4)} - 1 \right) + \frac{(1 + \lambda_3) \lambda_2}{2 \lambda_5} \left( 1 - \frac{\Gamma(n+1)\Gamma(n+1-i+\lambda_3)}{\Gamma(n+1-i)\Gamma(n+1+\lambda_3)} \right)
\]

The lambda values for the five-parameter lambda distribution are:

\[
\begin{align*}
\lambda_1 &= 11144,1 \\
\lambda_2 &= 810,001 \\
\lambda_3 &= 0,7 \\
\lambda_4 &= 0,17789 \\
\lambda_5 &= 0,400532
\end{align*}
\]

(11)

The computed value of \( \chi^2 \) equals 17.09 for which p-value is equal 0.1952. So, this time the model fits the data quite well.
IV. CONCLUSION

We considered the four- and five-parameter lambda distributions. We used two methods of estimating the distribution parameters: moment matching and least square ones. The empirical analysis shows that FLDP describes the stock market quotations better than $G_{\lambda}D$. Of course, the distribution found describe quotations only for a given period time. We do not claim that daily quotations of WIG20 index or Nikkei 225 index have probability distribution with parameters shown in Table 2 and 3. It is sometimes claimed in the literature that generalizations of Tukey-lambda distributions are not easy to use because it is difficult to find the distribution parameters. However, in this paper we show that $G_{\lambda}D$ and FLDP may be used to describe any asymmetric distribution.

REFERENCES

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